# Chapter 14

# Multiple integrals

In this chapter we study the **double integral** and **triple integral**. First, we define how to compute volumes of a solid by two methods: One by Carvalieri principle and the other by double integral. Then we show the relation between them. In fact, one can interpret the Carvalieri principle as an iterated integral and show this equals the double integral. The triple integral can be treated similarly.

## 14.1 Double integral over a rectangle

### Double Integral of a function

When $f(x, y)$ is a function over $R$, then the **double integral** of $f$ is the volume of the region above $R$ and under the graph of $f$. But the double integral of more general function (say continuous, or piecewise continuous) $f$ can be similarly defined.

**Definition 14.1.1.** Assume $R = \{(x, y)\colon a \le x \le b, c \le y \le d\}$. Then we Subdivide two intervals $[a, b]$, $[c, d]$ into $n$ -intervals

$$a = x_0 < x_1 < \cdots < x_n = b, \quad c = y_0 < y_1 < \cdots < y_n = d.$$

We call the subrectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ a **partition** of $R$ and let

$$\Delta x_i = x_i - x_{i-1}, \quad \Delta y_j = y_j - y_{j-1}.$$

**Definition 14.1.2.** Given any function $f$ defined on $R$, and for any point $c_{ij}$

in $R_{ij}$ consider the sum

$$S = \mathcal{R}(f) = \sum_{i,j=1}^{n} f(c_{ij})\Delta A_{ij}, \tag{14.1}$$

where $\Delta A_{ij} = \Delta x_i \Delta y_j$ is the area of $R_{ij}$. It is called **Riemann sum** of $f$ corresponding to the partition. Here $\|\mathcal{P}\| = \max_{i,j}\{\Delta x_i, \Delta y_j\}$ is called the **norm(size)** of the partition.

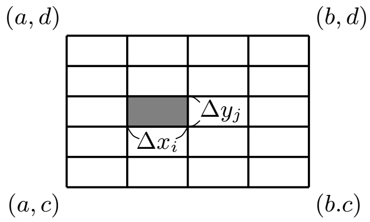



Figure 14.1: A partition of a rectangle

**Definition 14.1.3** (Double integral)**.** If the sum $S$ converge to the same limit regardless of the points $c_{ij}$ and regardless of the partition, then $f$ is called **integrable** over $R$ and we write its limit by

$$\iint_R f(x,y)\,dA = \lim_{\|\mathcal{P}\|\to 0} \sum_{i,j=1}^{n} f(c_{ij})\Delta x_i \Delta y_j.$$

These are also written as $\int_R f\,dA$ or $\iint_R f(x,y)\,dxdy$.

In particular, if $f(x) = 1$ we define the area of a closed, bounded plane region $R$ is given as

$$A = \iint_R dA.$$

## Double integral as Volumes

If $f(x)$ is nonnegative, we may interpret the double integral as the volume of a solid region over $R$ bounded by the surface $z = f(x,y)$.

In particular, if $f(x) = 1$ we define the area of a closed, bounded plane region $R$ is given as

$$A = \iint_R dA.$$

## Reduction to iterated integrals - Fubini's Theorem

Consider the volume of a solid under $f$ over $R = [a, b] \times [c, d]$ as in figure 14.2. The cross section along $x = x_0$ is the set given by $\{(x_0, y, z) | 0 \leq z \leq f(x_0, y),\ (c \leq y \leq d)\}$. The area of cross section is

$$A(x_0) = \int_c^d f(x_0, y)\, dy.$$

Hence by Cavalieri principle, the volume is

$$\int_a^b A(x)\, dx = \int_a^b \left[ \int_c^d f(x, y)\, dy \right] dx.$$

The expression on the right hand side is called an **iterated integral**. On the other hand, if we cut it by the plane $y = y_0$, then the volume becomes

$$\int_a^b A(y)\, dy = \int_c^d \left[ \int_a^b f(x, y)\, dx \right] dy.$$

Since these two values are equal,

$$\int_a^b \left[ \int_c^d f(x, y)\, dy \right] dx = \int_c^d \left[ \int_a^b f(x, y)\, dx \right] dy.$$

This is called **Fubini's theorem**.

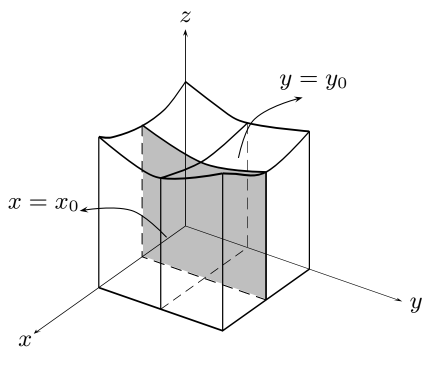



Figure 14.2: Fubini's theorem by Cavalieri Principle

**Example 14.1.4.** Evaluate

$$\iint_R (x^2+y^2)\,dxdy, \quad R=[-1,1]\times[0,1].$$

sol.

$$\int_0^1 \left[\int_{-1}^1 (x^2+y^2)dx\right] dy = \frac{4}{3}.$$

Now change the order to see the integrals are the same.

□

**Example 14.1.5.** Evaluate

$$\iint_S \cos x \sin y\,dxdy, \quad S=[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}].$$

sol.

$$\begin{aligned}\iint_S \cos x\sin y\,dxdy &= \int_0^{\pi/2}\left[\int_0^{\pi/2}\cos x\sin y\,dx\right]dy\\ &= \int_0^{\pi/2}\sin y\left[\int_0^{\pi/2}\cos x\,dx\right]dy = \int_0^{\pi/2}\sin y\,dy = 1.\end{aligned}$$

Now change the order to see the result is the same (skip).

□

**Theorem 14.1.6** (Fubini Theorem 1)**.** *Let $f$ be continuous on $R=[a,b]\times[c,d]$. Then $f$ satisfies*

$$\int_a^b\left[\int_c^d f(x,y)\,dy\right]dx = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy = \iint_R f(x,y)\,dA. \qquad (14.2)$$

Skip the proof.

**Example 14.1.7.** Find the volume of the region $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 2-x-y$.

sol. First fix $x$. Then the area of cross section with a plane perpendicular to $x$-axis is

$$A(x)=\int_0^1 (2-x-y)\,dy.$$

So the volume is

$$\begin{aligned} V &= \int_0^1 A(x)\,dx = \int_{x=0}^{x=1}\int_{y=0}^{y=1} (2-x-y)\,dydx \\ &= \int_0^1 \left[2y - xy - \frac{y^2}{2}\right]_0^1 dx \\ &= \int_0^1 \left(\frac{3}{2} - x\right) dx = \left[\frac{3x}{2} - \frac{x^2}{2}\right]_0^1 = 1. \end{aligned}$$

Change the order: You can fix $y$. Then the area of cross section with a plane perpendicular to $y$-axis is

$$A(y) = \int_0^1 (2-x-y)\,dx.$$

Hence the volume is

$$\begin{aligned} V &= \int_0^1 A(y)\,dy = \int_{y=0}^{y=1}\int_{x=0}^{x=1} (2-x-y)\,dxdy \\ &= \int_0^1 \left[2x - \frac{x^2}{2} - xy\right]_0^1 dy \\ &= \int_0^1 \left(\frac{3}{2} - y\right) dy = \left[\frac{3y}{2} - \frac{y^2}{2}\right]_0^1 = 1. \end{aligned}$$

□

**Example 14.1.8.** Compute $\iint_R (x^2+y)dA$, where $A = [0,1]\times[0,1]$.

**sol.**

$$\iint_R (x^2+y)dA = \int_0^1\int_0^1 (x^2+y)dxdy = \int_0^1 [\int_0^1 (x^2+y)dx]dy = \int_0^1 (\frac{1}{3}+y)dy = \frac{5}{6}.$$

□

**Example 14.1.9.** Find $\iint_R f(x,y)\,dxdy$. Here the function $f = y(x^3-12x)$ takes both positive and negative values and $R$ is given by $-2 \le x \le 1$, $0 \le y \le 1$.

**sol.**

$$\iint_R y(x^3-12x)dxdy = \int_0^1 \left[\int_{-2}^1 y(x^3-12x)dx\right] dy = \frac{57}{4}\int_0^1 ydy = \frac{57}{8}.$$

□

## 14.2 Double integral over general regions

So far we have defined double integral over a rectangle. How can we define double integral on general domains?

Now we define the integral of more general functions.

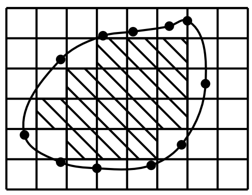


Figure 14.3: Partitioning of nonrectangular region

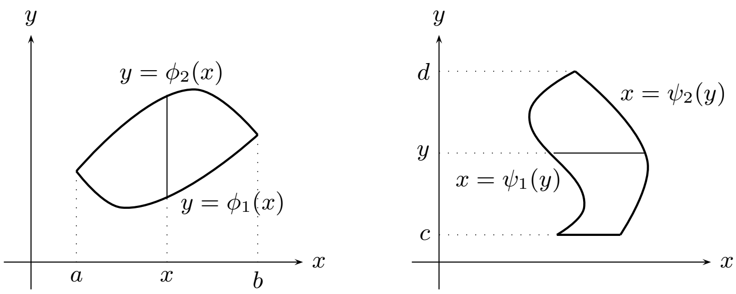
We proceed as befre, but only count the sub-rectangle completely contained in the region:

$$S = \mathcal{R}(f) = \sum f(c_{ij})\Delta A_{ij}, \tag{14.3}$$

where the sum is taken over subrectangles completely contained in the region. If this limit exists as $\|\mathcal{P}\| \to 0$ we define it as a double integral.

For computational purpose, we classify the regions.

**Definition 14.2.1. Elementary regions**




(a) region of type 1 (b) region of type 2

Figure 14.4: region of type 1, region of type 2

There are three kind of elementary regions: Let $y = \phi_1(x)$, $y = \phi_2(x)$ be two continuous functions satisfying $\phi_1(x) \le \phi_2(x)$ for $x \in [a, b]$. Then the region

$$D = \{(x, y) \mid a \le x \le b,\ \phi_1(x) \le y \le \phi_2(x)\}$$

is called **region of type 1**.

Now change the role of $x$, $y$ as in figure 14.4 (b). If $x = \psi_1(y)$, $x = \psi_2(y)$, satisfies $\psi_1(y) \le \psi_2(y)$ for $y \in [c, d]$, then the region determined by

$$D = \{(x, y) \mid c \le y \le d,\ \psi_1(y) \le x \le \psi_2(y)\}$$

is called **region of type 2**. The region that is both Type 1 and Type 2 is called **region of type 3**. These are called **elementary regions**.

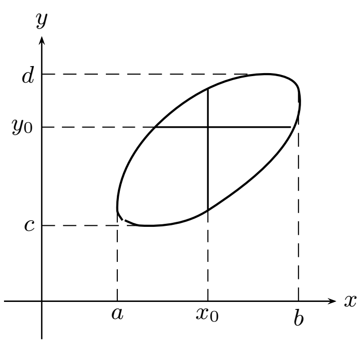



Figure 14.5: Region of type 3

## Properties of integral

**Theorem 14.2.2.** *Let $f$, $g$ be integrable over $R$, $R_1$, $R_2$. Then we have*

*(1)* $\displaystyle\iint_R cf(x,y)\,dxdy = c\iint_R f(x,y)\,dxdy$, *( $c$ is constant).*

*(2)* $\displaystyle\iint_R (f(x,y) + g(x,y))\,dxdy$
$\displaystyle\qquad = \iint_R f(x,y)\,dxdy + \iint_R g(x,y)\,dxdy.$

*(3)* *If* $\displaystyle f(x,y) \ge 0, \iint_R f(x,y)\,dxdy \ge 0.$

*(4)* *If* $\displaystyle f(x,y) \ge g(x,y), \iint_R f(x,y)\,dxdy \ge \iint_R g(x,y)\,dxdy.$

(5) *If $R_1$ and $R_2$ do not meet, then for $R = R_1 \cup R_2$*

$$\iint_R f(x,y)\,dxdy = \iint_{R_1} f(x,y)\,dxdy + \iint_{R_2} f(x,y)\,dxdy.$$

(6) $\left|\iint_R f dA\right| \le \iint_R |f| dA.$

## Integrals over elementary regions(by extension to 0)

Now we are ready to define the integral of $f$ defined on an elementary region. The idea is to extend the function to a rectangular domain. Given a continuous function $f$ on $D$ where $D$ is an elementary region

$$D = \{(x,y) \mid \phi_1(x) \le y \le \phi_2(x), \quad a \le x \le b\},$$

we consider a rectangle which contains $D$ and extend $f$ to $R$ outside $D$ by zero:

$$f^{ext}(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in R \setminus D. \end{cases}$$

Then $f^{ext}$ has discontinuities on the graphs of $y = \phi_1(x)$, $y = \phi_2(x)$, $a \le x \le b$. Hence it is integrable by Theorem **??**. Now we can define the integral of $f$ over $R$.

**Definition 14.2.3.** The integral of $f$ is defined as

$$\iint_D f(x,y)\,dA := \iint_R f^{ext}(x,y)\,dA.$$

From this definition we have an important result useful in the computation of double integral.

**Theorem 14.2.4** (Fubini's Theorem (Stronger form))**.** *Let $f$ be a continuous on an elementary region $D \subset R$.*

(1) *If $D$ is a domain of type 1, i.e, $D = \{(x,y) : \phi_1(x) \le y \le \phi_2(x),\ a \le x \le b\}$ for some continuous functions $\phi_1, \phi_2$, then $f$ is integrable on $D$ and*

$$\iint_D f(x,y)\,dA = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\right] dx.$$

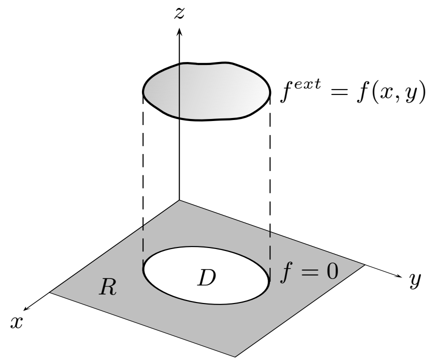



Figure 14.6: Extension of a function

*(2) Similarly if $D$ is a domain of type 2, i.e, $D = \{(x, y) : \psi_1(y) \le x \le \psi_2(y),\ c \le y \le d\}$ for some continuous functions $\psi_1, \psi_2$, then*

$$\iint_D f(x, y)\, dA = \int_c^d \left[ \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx \right] dy.$$

*Proof.* By Fubini theorem, we have

$$\begin{aligned} \iint_D f(x, y)\, dA &= \iint_D f^{ext}(x, y)\, dA && (14.4) \\ &= \int_a^b \int_c^d f^{ext}(x, y)\, dydx && (14.5) \\ &= \int_c^d \int_a^b f^{ext}(x, y)\, dxdy. && (14.6) \end{aligned}$$

For type 1 region, we see

$$\int_c^d f^{ext}(x, y)\, dy = \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy.$$

Hence by (14.5) we obtain (1). For type 2 region, we see

$$\int_a^b f^{ext}(x, y)\, dx = \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx.$$

Hence by (14.5) we obtain (2). □

**Example 14.2.5.** Find the following integral when $D: 0 \le x \le 1,\ x \le y \le 1$

$$\iint_D (x + y^2)\, dxdy.$$

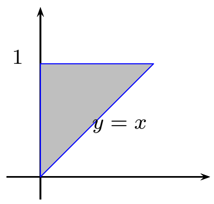



Figure 14.7: Region $0 \le x \le 1,\ x \le y \le 1$

sol. Use Fubini's theorem

$$\begin{aligned}\int_0^1 \int_x^1 (x + y^2)\, dydx &= \int_0^1 \left[xy + \frac{y^3}{3}\right]_x^1 dx \\ &= \int_0^1 \left(x + \frac{1}{3} - x^2 - \frac{x^3}{3}\right) dx \\ &= \left[\frac{x^2}{2} + \frac{x}{3} - \frac{x^3}{3} - \frac{x^4}{12}\right]_0^1 = \frac{5}{12}.\end{aligned}$$

□

**Example 14.2.6.** Find $\iint_D x^2 y\, dA$ where $D$ is given by $0 \le x,\ 3x^2 \le y \le 4 - x^2$. (Figure 14.8)

sol. Two curves $y = 3x^2$ and $y = 4 - x^2$ meet at the point $(1, 3)$. Hence the integral becomes

$$\begin{aligned}\int_0^1 \int_{3x^2}^{4-x^2} x^2 y\, dydx &= \int_0^1 \left(\frac{x^2y^2}{2}\right)\Big|_{y=3x^2}^{4-x^2} dx \\ &= \int_0^1 \left(\frac{x^2}{2}((4 - x^2)^2 - (3x^2)^2)\right) dx \\ &= \frac{1}{2}\int_0^1 x^2(16 - 8x^2 + x^4 - 9x^4)\, dx = \frac{136}{105}.\end{aligned}$$

□

**Example 14.2.7.** Find $\iint_D (x^3 y + \cos x)\, dA$ where $D$ is given by $0 \le x \le \pi/2$, $0 \le y \le x$.

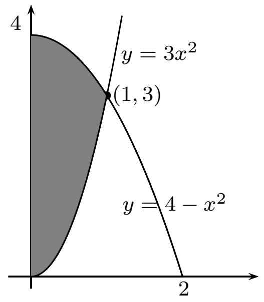



Figure 14.8: Domain of integration of example 14.2.6

**sol.**

$$\begin{aligned}&\iint_D (x^3y + \cos x)\, dA \\ &= \int_0^{\pi/2} \int_0^x (x^3y + \cos x) dy\, dx \\ &= \int_0^{\pi/2} \left[\frac{x^3y^2}{2} + y\cos x\right]_{y=0}^x dx = \int_0^{\pi/2} \left(\frac{x^5}{2} + x\cos x\right) dx \\ &= \frac{\pi^6}{768} + \frac{\pi}{2} - 1.\end{aligned}$$

□

**Example 14.2.8.** Find volume of tetrahedron bounded by the planes $x = 0, y = 0, z = 0, y - x + z = 1$. (Fig 14.9)

**sol.** We let $z = f(x, y) = 1 - y + x$. Then the volume of tetrahedra is the volume under the graph of $f$. Hence

$$\begin{aligned}\iint_D (1 - y + x) dA &= \int_{-1}^0 \int_0^{1+x} (1 - y + x) dy dx \\ &= \int_{-1}^0 \left[(1 + x)y - \frac{y^2}{2}\right]_{y=0}^{1+x} dx = \frac{1}{6}.\end{aligned}$$

□

**Example 14.2.9.** Let $D$ be given by $D = \{(x, y)|0 \le x \le \ln 2, \quad 0 \le y \le e^x - 1\}$. Express the double integral

$$\iint_D f(x, y)\, dA$$

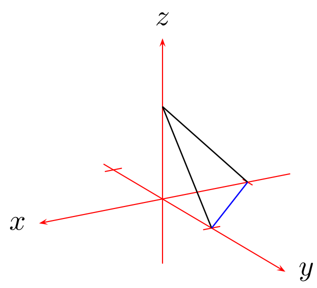



Figure 14.9: Tetrahedra $z = x - y + 1$ of

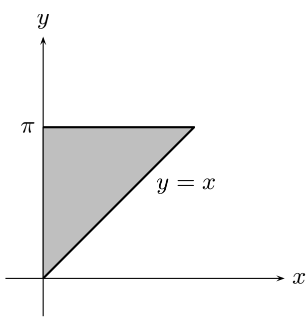



Figure 14.10: $0 \leq x \leq \pi$, $x \leq y \leq \pi$

in two iterated integrals.

**sol.** See figure 14.11. To view it as a region of type 1, the points of intersection is $y = 0$, $y = e^x - 1 (0 \leq x \leq \ln 2)$. Hence

$$\int_0^{\ln 2} \int_0^{e^x - 1} f(x, y)\, dy dx.$$

As a $y$-simple region, the points of intersection is $x = \ln(y + 1)$, $x = \ln 2 (0 \leq y \leq 1)$. So the integral is

$$\int_0^1 \int_{\ln(y+1)}^{\ln 2} f(x, y)\, dx dy.$$

□

**Example 14.2.10.** Given domain $D$ (Fig. 14.12) by

$$4-2x \le y \le 4-x^2, \quad 0 \le x \le 2.$$

Find

$$\iint_D (1+x)\,dA.$$

**sol.** This region is of third kind.

$$\begin{aligned}\int_0^2 \int_{4-2x}^{4-x^2} (1+x)\,dydx &= \int_0^2 [(1+x)y]_{y=4-2x}^{y=4-x^2}\,dx \\ &= \int_0^2 (-x^3+x^2+2x)\,dx \\ &= \left[-\frac{x^4}{4}+\frac{x^3}{3}+x^2\right]_0^2 = \frac{8}{3}.\end{aligned}$$

On the other hand, as a function of $y$ $x=(4-y)/2$, $x=\sqrt{4-y}$. So

$$\begin{aligned}\int_0^4 \int_{(4-y)/2}^{\sqrt{4-y}} (1+x)\,dxdy &= \int_0^4 \left[x+\frac{x^2}{2}\right]_{x=(4-y)/2}^{x=\sqrt{4-y}} dy \\ &= \int_0^4 \left(\sqrt{4-y}-\frac{(4-y)^2}{8}\right) dy \\ &= \left[-\frac{2}{3}(4-y)^{3/2}+\frac{(4-y)^3}{24}\right]_0^4 \\ &= \frac{2}{3}4^{3/2}-\frac{4^3}{24}=\frac{8}{3}.\end{aligned}$$

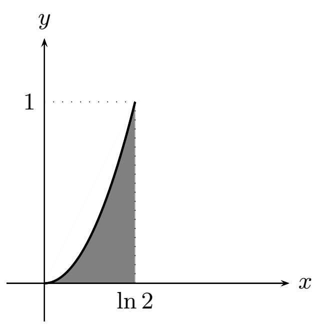



Figure 14.11: $0 \le y \le e^x - 1, \quad 0 \le x \le \ln 2$

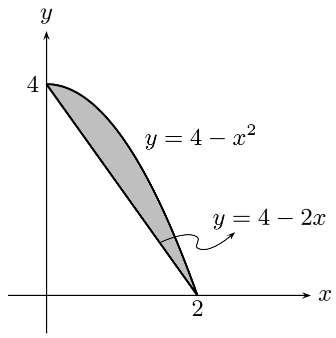



Figure 14.12: $4 - 2x \le y \le 4 - x^2$

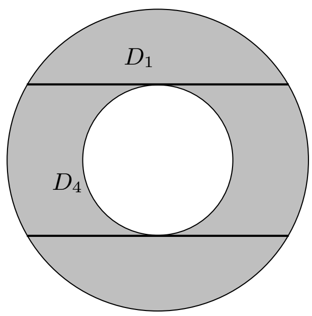



Figure 14.13: Region can be divided

□

**Example 14.2.11** (Breaking into several pieces)**.**

$$\int_D f\,dA = \int_{D_1} f\,dA + \int_{D_2} f\,dA + \int_{D_3} f\,dA + \int_{D_4} f\,dA.$$

See Figure 14.13

## 14.3 Area by double integral and Change order of integration

Suppose $D$ is of type 3. Then it is given by two ways:

$$\phi_1(x) \le y \le \phi_2(x),\ a \le x \le b$$

and

$$\psi_1(y) \le x \le \psi_2(y),\ \ c \le y \le d.$$

Thus by Theorem 14.2.4

$$\iint_D f(x,y)dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dydx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dxdy.$$

**Example 14.3.1.** Compute by change of order of integration

$$\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2-y^2)^{1/2}\,dydx.$$

**sol.**

$$\begin{aligned}\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2-y^2)^{1/2}\,dydx &= \int_0^a \int_0^{(a^2-y^2)^{1/2}} (a^2-y^2)^{1/2}\,dxdy \\ &= \int_0^a [x(a^2-y^2)^{1/2}]_0^{(a^2-y^2)^{1/2}} (a^2-y^2)^{1/2}\,dy \\ &= \int_0^a (a^2-y^2)\,dy = \frac{2a^3}{3}.\end{aligned}$$

□

There are cases when the given integral is almost impossible to find, but if we change the order the integral can be found.

**Example 14.3.2.** Find

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y}\,dydx.$$

**sol.** It is not easy to find the integral as the given form. But if we change the order of integration (fig **??**)

$$\begin{aligned}\int_0^\pi \int_x^\pi \frac{\sin y}{y}\,dydx &= \int_0^\pi \int_0^y \frac{\sin y}{y}\,dxdy \\ &= \int_0^\pi \left[\frac{\sin y}{y}x\right]_{x=0}^{x=y} dy \\ &= \int_0^\pi \sin y\,dy = [-\cos y]_0^\pi = 2.\end{aligned}$$

□

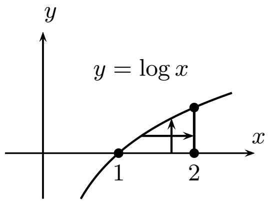



Figure 14.14: Change order of integration

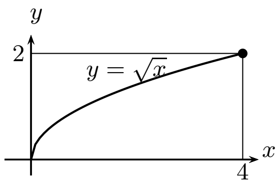



Figure 14.15: Change order of integration

**Example 14.3.3.** Find

$$\int_0^1 \int_y^1 \frac{e^x - 1}{x}\, dx\, dy.$$

**Example 14.3.4.** Find

$$\int_0^2 \int_{y^2}^4 y\cos(x^2)\, dx\, dy.$$

**sol.** It is very difficult to find $\int_{y^2}^4 \cos(x^2)\, dx$. However, if we change the order of integration to have (Figure 14.15)

$$\begin{aligned}\int_0^2 \int_{y^2}^4 \cos(x^2)\, dx\, dy &= \int_0^4 \int_0^{\sqrt{x}} y\cos(x^2)\, dy\, dx \\ &= \int_0^4 \frac{y^2}{2}\cos(x^2)\Big|_0^{\sqrt{x}}\, dx \\ &= \int_0^4 \frac{x}{2}\cos(x^2)\, dx \\ &= \frac{1}{4}\int_0^{16} \cos u\, du = \frac{1}{4}\sin 16.\end{aligned}$$

□

### 14.3.1 Average value

**Theorem 14.3.5.** *Suppose $f : D \to \mathbb{R}$ is continuous on a region $D$ and $m = \min_D f(x, y) \le f(x, y) \le M = \max_D f(x, y)$. Then we have*

$$m \le \frac{1}{A(D)} \iint_D f\, dA \le M. \tag{14.7}$$

**Example 14.3.6.** Estimate

$$\int_D \frac{1}{\sqrt{1 + x^6 + y^7}} dxdy$$

where $D$ is the unit square. Then we can easily see the following holds.

$$\frac{1}{\sqrt{3}} \le \frac{1}{\sqrt{1 + x^6 + y^7}} \le 1.$$

**Theorem 14.3.7.** *If $f$ is continuous over a closed, bounded region $D$, then there is a point $(x_0, y_0) \in D$ such that*

$$f(x_0, y_0) = \frac{1}{A(D)} \iint_D f.$$

## 14.4 Double integral in polar coordinate form

Sometimes a domain is given in polar coordinate. In this case the double integral is easier to perform in polar coordinate. We are given a region $D$ by

$$D = \{(r, \theta) \mid \phi_1(\theta) \le r \le \phi_2(\theta), \quad \alpha \le \theta \le \beta\}.$$

Its boundary is described by the curves $r = \phi_1(\theta)$, $r = \phi_2(\theta)$ and $\theta = \alpha, \theta = \beta$. We divide $D$ by the curves $r =$ constant and the lines given by $\theta =$ constant(fig 14.16), i.e., with $\Delta r = r/m$, $\Delta\theta = (\beta - \alpha)/l$, we have

$$r_0 = \Delta r,\ r_1 = 2\Delta r, \ldots, r_{m+1} = m\Delta r,$$

and

$$\theta_0 = \alpha,\ \theta_1 = \alpha + \Delta\theta,\ \ldots,\ \theta_{l+1} = \alpha + l\Delta\theta = \beta.$$

Each small curved rectangle shaped region is called a **polar rectangle**. It is obtained by subtracting the inner sector from the outer sector (Fig. 14.17).

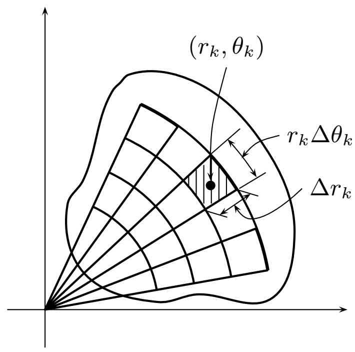



Figure 14.16: Partition in polar coordinate

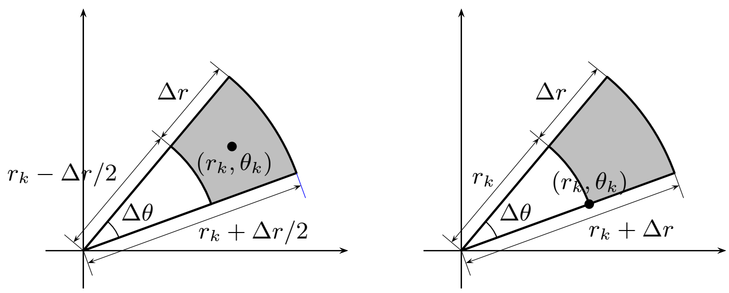



Figure 14.17: Area of polar sector

We label them as $\Delta A_1, \ldots, \Delta A_n$.

Choose any point $(r_k, \theta_k)$ in $\Delta A_k$ and consider the Riemann sum

$$\mathcal{R}(f, n) = S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\Delta A_k.$$

Let $\delta = \max_{i,j}\{\Delta r_i, \Delta\theta_j\}$. If the limit $\lim_{n\to\infty} \mathcal{R}(f, n)$ exists (as $\delta$ approaches 0), then it is defined as the integral of $f$ on $D$ and we write

$$\iint_D f(r, \theta)\, dA.$$

How to evaluate the integral $\iint_D f(r, \theta)\, dA$? For convenience assume the point $(r_k, \theta_k)$ is at the center of $\Delta A_k$(Figure 14.17, left). The area of $\Delta A_k$ is

$$\frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \Delta\theta - \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \Delta\theta = r_k \Delta r \Delta\theta.$$

Another way: If we let $r_k$ denote the inner radius, the area of outer sector minus inner sector(figure 14.17, right) is

$$\frac{1}{2}(r_k + \Delta r)^2 \Delta\theta - \frac{1}{2} r_k^2 \Delta\theta = r_k \Delta r \Delta\theta + \frac{1}{2}\Delta r^2 \Delta\theta.$$

Hence the Riemann sum is

$$\mathcal{R}(f, n) = (\text{double sum}) \sum_{k=1}^{n} f(r_k, \theta_k) r_k \Delta r \Delta\theta + O(\Delta r^2 \Delta\theta).$$

As $n \to \infty$, this approaches $\iint_D f(r,\theta) r\, dr d\theta$.

**Proposition 14.4.1.** *If $D$ is given by $D = \{(r,\theta) \mid \phi_1(\theta) \le r \le \phi_2(\theta), \alpha \le \theta \le \beta\}$, the integral of $f$ can be evaluated as the iterated integral:*

$$\iint_D f(r,\theta)\, dA = \int_\alpha^\beta \int_{\phi_1(\theta)}^{\phi_2(\theta)} f(r,\theta) r\, dr d\theta.$$

**Example 14.4.2.** $D$ is the disk of radius $a$ about the origin. Find

$$\iint_D e^{x^2+y^2}\, dA.$$

**sol.** Use polar coordinate $x^2 + y^2 = r^2$, $dA = r dr d\theta$. Then

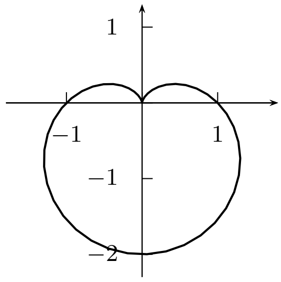
$$\int_0^{2\pi} \int_0^a e^{r^2} r\, dr d\theta = \int_0^{2\pi} \left[\frac{e^{r^2}}{2}\right]_{r=0}^{r=a} d\theta = \pi(e^{a^2} - 1).$$

□

**Example 14.4.3.** Find the area of the region inside the cardioid $r = 1 - \sin\theta$.

**sol.** Refer to Fig 14.18 for the cardioid. We see $0 \le r \le 1 - \sin\theta$

$$\begin{aligned}
\int_0^{2\pi} \int_{r=0}^{r=1-\sin\theta} r\, dr d\theta &= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_{r=0}^{r=1-\sin\theta} d\theta \\
&= \int_0^{2\pi} \frac{(1-\sin\theta)^2}{2}\, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (1 - 2\sin\theta + \sin^2\theta)\, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (1 - 2\sin\theta + \frac{1-\cos 2\theta}{2})\, d\theta
\end{aligned}$$




$r = 1 - \sin\theta$

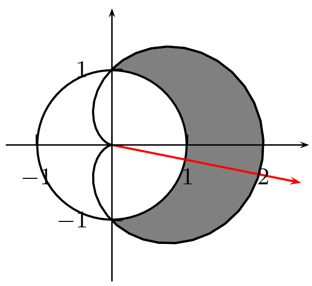
Figure 14.18: $r = 1 - \sin\theta$

$$= \frac{1}{2}\left[\theta + 2\cos\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$
$$= \frac{3}{2}\pi.$$

□

□

**Example 14.4.4.** The area inside of the cardioid $r = 1 + \cos\theta$ and outside of the unit circle $r = 1$.




$r = 1 + \cos\theta$

Figure 14.19: Find the limits of integral $r = 1$, $r = 1 + \cos\theta$

**Example 14.4.5.** Find the area of the region bounded by the curve $r = 1 + 2\cos\theta$ except the smaller one.

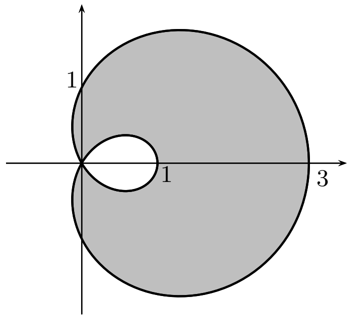



Figure 14.20: $r = 1 + 2\cos\theta$

**sol.** This graph is symmetric about $x$-axis. When $\theta = 0$, it passes $(3, 0)$, and when $\theta = \pi/2$ it passes $(0, 1)$, when $\theta = 2\pi/3$, it passes the origin, when $\theta = \pi$, passes $(1, 0)$. Hence the graph is as figure 14.20. Hence the area bounded by the curve is

$$\begin{aligned}
2\int_0^{2\pi/3}\int_0^{1+2\cos\theta} r\,dr d\theta &= \int_0^{2\pi/3} \left[r^2\right]_0^{1+2\cos\theta} d\theta \\
&= \int_0^{2\pi/3} (1 + 4\cos\theta + 4\cos^2\theta)\, d\theta \\
&= [\theta + 4\sin\theta + 2\theta + \sin 2\theta]_0^{2\pi/3} \\
&= 2\pi + \frac{3\sqrt{3}}{2}.
\end{aligned}$$

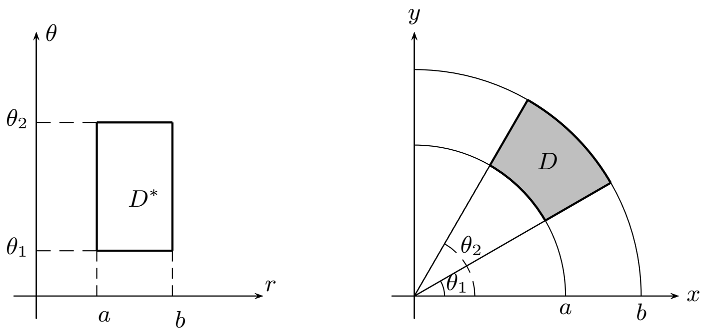
Meanwhile the area inside the inner curve is

$$\begin{aligned}
2\int_{2\pi/3}^{\pi}\int_0^{1+2\cos\theta} r\,dr d\theta &= \int_{2\pi/3}^{\pi} \left[r^2\right]_0^{1+2\cos\theta} d\theta \\
&= \int_{2\pi/3}^{\pi} (1 + 4\cos\theta + 4\cos^2\theta)\, d\theta \\
&= [\theta + 4\sin\theta + 2\theta + \sin 2\theta]_{2\pi/3}^{\pi} \\
&= \pi - \frac{3\sqrt{3}}{2}.
\end{aligned}$$

So the desired area is $(2\pi + 3\sqrt{3}/2) - (\pi - 3\sqrt{3}/2) = \pi + 3\sqrt{3}$.

□

More generally, we obtain (Need to explain the mapping $T$)




**Example 14.4.6.** Change the integral $\iint f(x,y)\,dxdy$ to polar coordinate.

**sol.** Since $x = r\cos\theta$, $y = r\sin\theta$, we can let $T(r,\theta) = (r\cos\theta, r\sin\theta)$. Then Jacobian is

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Hence

$$\iint f(x,y)\,dxdy = \iint f(r\cos\theta, r\sin\theta)\,r dr d\theta.$$

□

**Example 14.4.7.** $D$ is between two concentric circles: $x^2+y^2=4$, $x^2+y^2=1(x,y\ge 0)$. Find the integral

$$\iint_D \sqrt{x^2+y^2+1}\,dxdy.$$

Here $D$ is the quoter of the annulus $\sqrt{1-x^2} \le y \le \sqrt{4-x^2}$.

**sol.** Use polar coordinate. We see the domain of integration in $(r,\theta)$ is

$$D^* = \{(r,\theta) | 1 \le r \le 2, 0 \le \theta \le \pi/2\}.$$

$$\begin{aligned}
\iint_D \sqrt{x^2+y^2+1}\,dxdy &= \iint_{D^*} \sqrt{r^2+1}r\,drd\theta \\
&= \int_0^{\pi/2}\int_1^2 \frac{1}{2}\sqrt{r^2+1}(2r)drd\theta \\
&= \int_0^{\pi/2} \frac{1}{3}(r^2+1)^{3/2}|_1^2 d\theta \\
&= \int_0^{\pi/2} \frac{1}{3}(5^{3/2}-2^{3/2})d\theta = \frac{\pi}{6}(5^{3/2}-2^{3/2}).
\end{aligned}$$

□

**Example 14.4.8.** $D$ is the region between two concentric circles in the first quadrant: $1 \le x^2+y^2 \le 4$, $(x, y \ge 0)$. Find the integral

$$\iint_D \log(x^2+y^2)dxdy.$$

**sol.** Use polar coordinate. Since the boundary of the region are described by $r = 1, 2$, $0 \le \theta \le \pi/2$, we let $D^* = [1,2]\times[0,\pi/2]$ and $T(r,\theta) = (r\cos\theta, \sin\theta)$. Then $T(D^*) = D$ and

$$\begin{aligned}
\iint_D \log(x^2+y^2)dxdy &= \iint_{D^*} (\log r^2) r dr d\theta \\
&= \int_1^2\int_0^{\pi/2} 2r\log r d\theta dr \\
&= \int_1^2 \pi r \log r dr \\
&= \pi\left[\frac{r^2}{2}\log r - \frac{r^2}{4}\right]_1^2 \\
&= \pi(2\log 2 - \frac{3}{4}).
\end{aligned}$$

□

**Example 14.4.9** (The Gaussian integral)**.** Show that

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}.$$

To compute this, let us first observe

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 &= \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy \\ &= \lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}dxdy.\end{aligned}$$

Thus it is necessary to compute

$$\iint_{D_a} e^{-(x^2+y^2)}dxdy.$$

By

$$\begin{aligned}\iint_{D_a} e^{-(x^2+y^2)}dxdy &= \int_0^{2\pi}\int_0^a e^{-r^2}r\,drd\theta = \int_0^{2\pi}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^a \\ &= -\frac{1}{2}\int_0^{2\pi}(e^{-a^2}-1)d\theta = \pi(1-e^{-a^2}).\end{aligned}$$

Let $a \to \infty$. Then we obtain the result.

**Example 14.4.10.** Evaluate

$$\iint_R e^{x^2+y^2}dxdy,$$

where $R$ is the semi circular region $0 \le y \le \sqrt{1-x^2}$.

To compute this, we see

$$\begin{aligned}\iint_R e^{x^2+y^2}dxdy &= \int_0^{\pi}\int_0^1 e^{r^2}rdrd\theta \\ &= \int_0^{\pi}\frac{1}{2}(e-1)d\theta \\ &= \frac{\pi}{2}(e-1).\end{aligned}$$

## 14.5 Triple integrals in rectangular coordinates

**Definition 14.5.1.** Assume $D = [a,b]\times[c,d]\times[p,q]$ be a box. Then we subdivide intervals $[a,b]$, $[c,d]$ and $[p,q]$ into $n$ -intervals

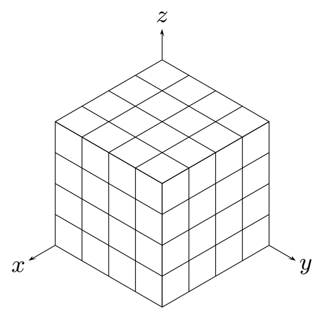



Figure 14.21: partition of a box region

$$
\begin{aligned}
a &= x_0 < x_1 < \cdots < x_n = b, \\
c &= y_0 < y_1 < \cdots < y_n = d, \\
p &= z_0 < z_1 < \cdots < z_n = q,
\end{aligned}
$$

and call the resulting subboxes $D_{jk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ a **partition** of $D$.

**Definition 14.5.2.** We let

$$\Delta x_i = x_i - x_{i-1}, \quad \Delta y_j = y_j - y_{j-1} \text{ and } \Delta z_k = z_k - z_{k-1}.$$

We partition the box into small $n^3$- subboxes as in Fig 14.21, and denote the volume of each subbox as $\Delta V_{ijk}$ $(i, j, k = 1, \dots, n)$ and let $\|\mathcal{P}\| = \max_{i,j,k}\{\Delta x_i, \Delta y_j, \Delta z_k\}$. Then the Riemann sum becomes

$$\mathcal{R}(f, n) = S_n = \sum_{i,j,k=1}^{n} f(c_{ijk}) \Delta V_{ijk}.$$

Here $c_{ijk}$ is any point in the subbox $D_{ijk}$.

**Definition 14.5.3.** If $\lim_n S_n = S$ exists independently of the choice of $c_{ijk}$, then we say $f$ is integrable in $D$ and call $S$ the **triple integral** and we write

$$\iiint_D f dV, \quad \iiint_D f(x, y, z) dV, \text{ or } \quad \iiint_D f(x, y, z) dx dy dz.$$

### Reduction to iterated integral

**Theorem 14.5.4** (Fubini's theorem). *Suppose $f$ is continuous on $D = [a, b] \times [c, d] \times [p, q]$. Then the triple integral $\iiint_D f(x, y, z)dxdydz$ equals with any of the following integrals.*

$$\int_p^q \int_c^d \int_a^b f(x, y, z)\, dxdydz, \quad \int_p^q \int_a^b \int_c^d f(x, y, z)\, dydxdz$$
$$\int_a^b \int_c^d \int_p^q f(x, y, z)\, dzdydx, \ \ etc.$$

**Example 14.5.5.**

$$\iiint_D e^{x+y+z} dV,$$

where $D$ is the unit cube at origin.

### Elementary regions

[sketch the region]

**Definition 14.5.6.** A region $D$ is **elementary regions** if the points lie between graph of continuous functions of two variables, and the domain of these functions is elementary. If $f$ is continuous on $D$, then we extend $f$ on a box $E$ containing $D$

$$f^{ext}(x, y, z) = \begin{cases} f(x, y, z), & (x, y, z) \in D \\ 0, & (x, y, z) \in E \setminus D \end{cases}$$

and define

$$\int_D f dV \equiv \iiint_D f dV = \iiint_E f^{ext} dV.$$

If $f = 1$, the volume of $D$ is defined as

$$V = \iiint_D dV.$$

Suppose $R$ is an elementary region in $xy$-plane and there are continuous functions $\gamma_1(x, y)$, $\gamma_2(x, y)$ such that

$$D = \{(x, y, z) \mid \gamma_1(x, y) \le z \le \gamma_2(x, y), \quad (x, y) \in R\}. \tag{14.8}$$

Then this is called an **elementary region of type 1**.

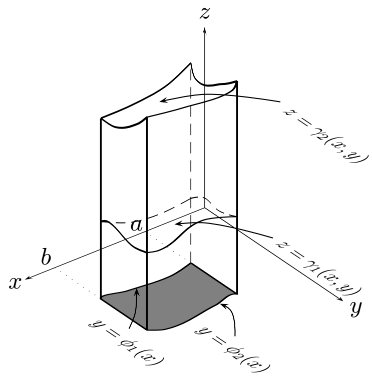



Figure 14.22: elementary region of type 1

If roles of $x, z$ are interchanged, i.e,

$$D = \{(x, y, z) \mid \gamma_1(y, z) \leq x \leq \gamma_2(y, z), \quad (y, z) \in R\} \tag{14.9}$$

for some elementary region $R$ in $(y, z)$-plane, then it is called an **elementary region of type 2**.

Similarly, we can define an **elementary region of type 3** and an **elementary region of type 4**.

**Example 14.5.7.** Describe the unit ball as an elementary region.

**sol.** The domain of defining function is described by

$$-\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}, \quad -1 \leq x \leq 1$$

while the functions are

$$-\sqrt{1 - x^2 - y^2} \leq z \leq \sqrt{1 - x^2 - y^2} \quad \text{on the unit disk.}$$

□

## Integrals over elementary regions

Suppose $D$ is defined by

$$D = \{(x, y, z) \mid \gamma_1(x, y) \leq z \leq \gamma_2(x, y), \quad (x, y) \in R\},$$

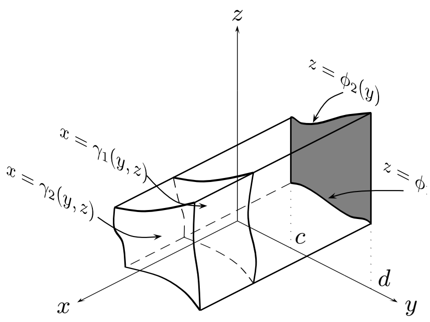



Figure 14.23: elementary region of 2

where $R$ is a type 1 region in $xy$-plane

$$R = \{(x, y) \mid \phi_1(x) \leq y \leq \phi_2(x), \quad a \leq x \leq b\}.$$

Then the integral is given by

$$\begin{aligned}\iiint_D f\,dV &= \iint_R \int f(x,y,z)\,dzdA \\ &= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dzdydx.\end{aligned}$$

**Example 14.5.8.** Find the volume of radius 1.

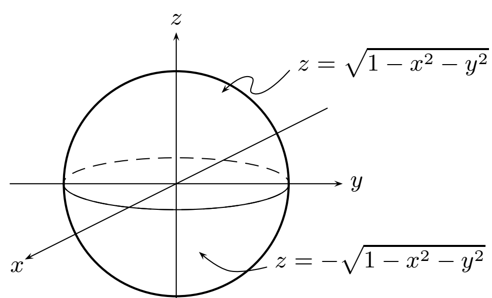



Figure 14.24: $x^2 + y^2 + z^2 = 1$

**sol.** Unit ball is described by $x^2+y^2+z^2 \le 1$. The volume is (Figure 14.24)

$$\int_D 1\,dV, \quad D = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}.$$

Here we can take $R = \{(x,y) \mid x^2+y^2 \le 1\}$ and $D = \{-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}, (x,y) \in R\}$. Hence

$$\begin{aligned}\iint_R \int dzdydx &= \iint_R \int_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} 1\,dzdydx \\ &= 2\int_R \sqrt{1-x^2-y^2}\,dydx \\ &= 2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dydx.\end{aligned}$$

Let $\sqrt{1-x^2} = a$. The inner integral is area of semi circle or radius $a$

$$2\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy = 2\int_{-a}^{a} \sqrt{a^2-y^2}dy = a^2\pi = (1-x^2)\pi.$$

Hence

$$\begin{aligned}2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dydx &= \int_{-1}^{1}(1-x^2)\pi\,dx \\ &= \left[(x-\frac{x^3}{3})\pi\right]_{-1}^{1} = 2(1-\frac{1}{3})\pi = \frac{4}{3}\pi.\end{aligned}$$

□

Other type of elementary regions can be described similarly. If a region can be described in all three ways we call these regions **symmetric elementary regions**.

**Example 14.5.9.** Let $D$ be the region bounded by $x+y+z=1$, $x=0$, $y=0$, $z=0$. Find

$$\iiint_D (1+2z)dxdydz.$$

**sol.** Let $R = \{(x,y) \mid 0 \le y \le 1-x, \quad 0 \le x \le 1\}$. Then $D$ is described by

$$D = \{(x,y,z) \mid 0 \le z \le 1-x-y, \quad (x,y) \in R\}$$

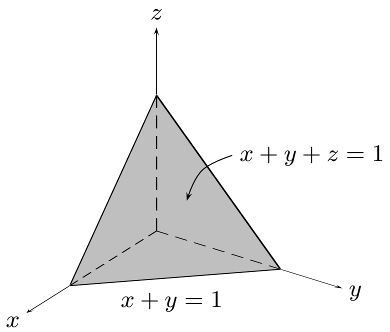



Figure 14.25: $x + y + z = 1$

and integrate along $z$ direction.

$$\begin{aligned}
\iiint_D (1+2z)\,dxdydz &= \iint_R \left[z+z^2\right]_0^{1-x-y}\,dxdy \\
&= \int_0^1 \int_{y=0}^{y=1-x} (1-x-y+(1-x-y)^2)\,dydx \\
&= \int_0^1 \left[-\frac{(1-x-y)^2}{2} - \frac{(1-x-y)^3}{3}\right]_{y=0}^{y=1-x} dx \\
&= \int_0^1 \left(\frac{(1-x)^2}{2} + \frac{(1-x)^3}{3}\right) dx = \frac{1}{4}.
\end{aligned}$$

□

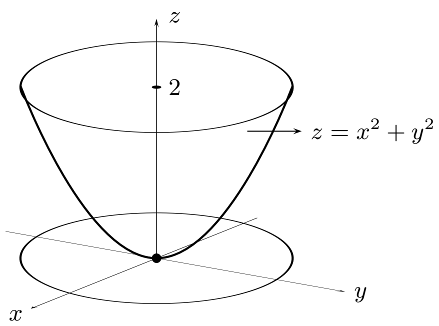



Figure 14.26: $z = x^2 + y^2$, $z = 2$

**Example 14.5.10.** Let $W$ be bounded by $x = 0, y = 0, z = 2$ and the surface $z = x^2 + y^2$ where $x \geq 0, y \geq 0$. Find $\iiint_W x\,dxdydz$.

**sol.** Method1. We describe the region by type 1.

$$0 \le x \le \sqrt{2}, \quad 0 \le y \le \sqrt{2-x^2}, \quad x^2+y^2 \le z \le 2.$$

$$\begin{aligned}\iiint_W x\,dxdydz &= \int_0^{\sqrt{2}} \left[\int_0^{\sqrt{2-x^2}} (\int_{x^2+y^2}^{2} x\,dz)dy\right] dx \\ &= \frac{8\sqrt{2}}{15}.\end{aligned}$$

Method2. We describe the region by type 2: Solving for $x$, i.e, $0 \le x \le (z-y^2)^{1/2}$, $(y,z) \in R$ where $R$ is given by the relation

$$0 \le z \le 2, \quad 0 \le y \le z^{1/2}.$$

Then

$$\begin{aligned}\iiint_W x\,dxdydz &= \iint_R \left(\int_0^{(z-y^2)^{1/2}} xdx\right) dydz \\ &= \int_0^2 \int_0^{z^{1/2}} \frac{z-y^2}{2} dydz \\ &= \int_0^2 \int_0^{z^{1/2}} \frac{z-y^2}{2} dydz \\ &= \int_0^2 \left.\frac{zy-\frac{y^3}{3}}{2}\right|_0^{z^{1/2}} dz \\ &= \int_0^2 \frac{z^3}{3} dz = \frac{8\sqrt{2}}{15}.\end{aligned}$$

□

**Example 14.5.11** (Example 1 p.911)**.** Find the volume of the region $D$ bounded by $z = x^2+3y^2$ and $z = 8-x^2-y^2$.

**sol.** We describe the region by type 1. First find the intersections of two surfaces. Set $x^2+3y^2 = 8-x^2-y^2$ to get $x^2+2y^2 = 4$. The the domain is the ellipse $x^2+2y^2 = 4$.

$$-2 \le x \le 2, \quad -\sqrt{(4-x^2)/2} \le y \le \sqrt{(4-x^2)/2}, \quad x^2+3y^2 \le z \le 8-x^2-y^2.$$

$$\begin{aligned} V(D) &= \iiint_D dzdxdy = \int_{-2}^{2} \left[ 2\int_0^{\sqrt{(4-x^2)/2}} (8-2x^2-4y^2)dy \right] dx \\ &= \int_{-2}^{2} \left[ 2(8-2x^2)y - \frac{4}{3}y^3 \right]_0^{\sqrt{(4-x^2)/2}} dx \\ &= 8\pi\sqrt{2}. \end{aligned}$$

□

**Example 14.5.12.** Evaluate

$$\int_0^1 \int_0^x \int_{x^2+y^2}^2 dzdydx.$$

**sol.** Sketch region first.

$$\begin{aligned} \int_0^1 \int_0^x \int_{x^2+y^2}^2 dzdydx &= \int_0^1 \int_0^x (2-x^2-y^2)dydx \\ &= \int_0^1 \int_0^x (2y - x^2y - \frac{y^3}{3})\Big|_0^x dx \\ &= \int_0^1 \int_0^x (2x - \frac{4x^3}{3})dx = x^2 - \frac{x^4}{3}\Big|_0^1 = \frac{2}{3}. \end{aligned}$$

□

**Example 14.5.13.** Find the common region of two cylinders (Figure 14.27) $x^2+y^2 \le 1$, $x^2+z^2 \le 1$ $(z \ge 0)$.

**sol.**

$$\begin{aligned} \iint_{x^2+y^2\le 1} \int_0^{\sqrt{1-x^2}} dzdxdy &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2}dydx \\ &= 2\int_{-1}^{1} (1-x^2)dx \\ &= 2\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 4(1-\frac{1}{3}) = \frac{8}{3}. \end{aligned}$$

□

**Example 14.5.14.** Find the region bounded by two paraboloid $z = x^2+y^2$ and $z = 2-3x^2-y^2$. (Figure 14.28)

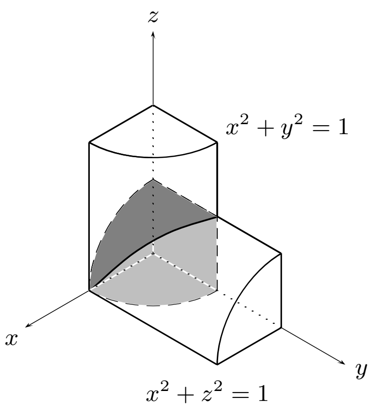



Figure 14.27: common region of two cylinders

**sol.** The intersection is the curve $x^2 + y^2 = 2 - 3x^2 - y^2$, i.e, $2x^2 + y^2 = 1$. If we let $R = \{(x, y) \colon 2x^2 + y^2 \le 1\}$ this region is 1st kind on $R$. Hence

$$\iiint_D dxdydz = \iint_{2x^2+y^2 \le 1} (2 - 3x^2 - y^2) - (x^2 + y^2)\, dxdy$$
$$= \iint_{2x^2+y^2 \le 1} (2 - 4x^2 - 2y^2)\, dxdy.$$

Now use polar coordinate $x = r\cos\theta/\sqrt{2}, y = r\sin\theta$. Then $dxdy = r/\sqrt{2}drd\theta$. Hence

$$\frac{1}{\sqrt{2}} \int_0^{2\pi} \int_{r \le 1} (2 - 2r^2) r\, drd\theta$$
$$= \frac{1}{\sqrt{2}} \int_0^{2\pi} \left[ r^2 - \frac{2r^4}{4} \right]_0^1 d\theta = \frac{\pi}{\sqrt{2}}.$$

□

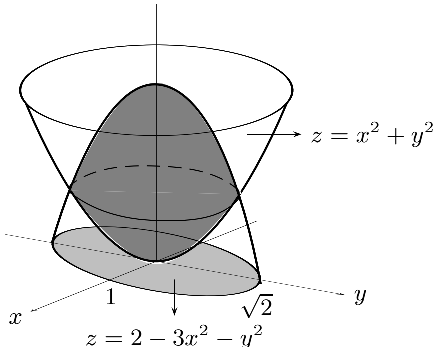



Figure 14.28: $z = x^2 + y^2$, $z = 2 - 3x^2 - y^2$

### Average

In $\mathbb{R}^n (n = 1, 2, 3)$, the average of a function $f$ defined on $I($ $D$ or $W)$is defined as

$$f_{av} = \frac{\int_a^b f(x)dx}{\int_a^b dx} = \frac{\int_a^b f(x)dx}{\text{length of } [a,b]}, \tag{14.10}$$

$$f_{av} = \frac{\iint_D f(x,y)dxdy}{\iint_D dA} = \frac{\iint_D f(x,y)dxdy}{\text{area of } D}, \tag{14.11}$$

$$f_{av} = \frac{\iiint_W f(x)dx}{\iiint_W dV} = \frac{\iiint_W f(x)dx}{\text{volume of } W}. \tag{14.12}$$

**Example 14.5.15.** Find average of $f(x,y) = x\sin^2(xy)$ over $D = [0,\pi] \times [0,\pi]$.

## 14.6 Mass, Moments and Center of Mass

### 14.6.1 Mass and Moments

### Moment in 1 D

When a material with density $\delta(x)$ is placed on an interval $I$ of the $x$- axis, then the mass on the part $[x, x + \Delta x]$ is (approx.) $\delta(x)\Delta x$. The total mass lying on the interval $I$ is $\approx \sum_i \delta(x_i)\Delta x_i$. Taking the limit,

$$M = \int_I \delta(x)dx$$

and the moment is

$$\int_I x\delta(x)\,dx.$$

We choose a point $\bar{x}$ so that the moment w.r.t $\bar{x}$ is zero.

$$\int_I (x-\bar{x})\delta(x)\,dx = 0 \Rightarrow \bar{x} = \frac{\int x\delta(x)\,dx}{\int \delta(x)\,dx}.$$

## Moment in 2 D

For 2-D, we have

**Definition 14.6.1** (Moment, center of mass)**.** Let $\delta(x,y)$ be the density of some material lying on a region $R$ in the plane. The mass of this material occupying the place $[x, x+\Delta x]\times[y, y+\Delta y]$ is $\approx \delta(x,y)\Delta x\Delta y$, and in the limit

$$\begin{aligned}
\text{The total mass } M &= \iint_R \delta(x,y)dxdy,\\
\text{The moment w.r.t } x\text{-axis } M_x &= \iint_R y\delta(x,y)\,dxdy,\\
\text{The moment w.r.t } y\text{-axis } M_y &= \iint_R x\delta(x,y)\,dxdy.
\end{aligned}$$

The **center of mass** $(\bar{x},\bar{y})$ is defined as

$$\begin{aligned}
\bar{x} = \frac{M_y}{M} &= \frac{\iint_R x\delta(x,y)\,dxdy}{\iint_R \delta(x,y)\,dxdy},\\
\bar{y} = \frac{M_x}{M} &= \frac{\iint_R y\delta(x,y)\,dxdy}{\iint_R \delta(x,y)\,dxdy}.
\end{aligned}$$

The center of mass is defined so that it satisfies

$$\begin{aligned}
M_{\bar{x}} &= \iint_R (x-\bar{x})\delta(x,y)\,dxdy = 0,\\
M_{\bar{y}} &= \iint_R (x-\bar{y})\delta(x,y)\,dxdy = 0.
\end{aligned}$$

**Example 14.6.2.** A solid body occupies the region between $y = x$, $y = x^2$. The density is given by $\delta(x,y) = x$. Find the mass and $M_x$.

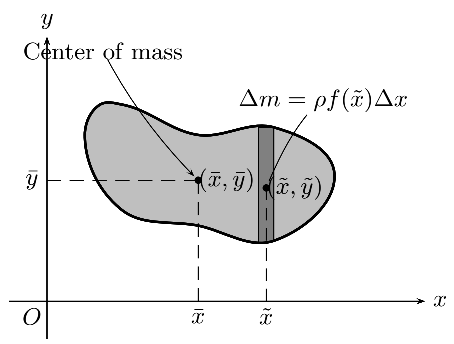



Figure 14.29: Vertical strip of mass $\Delta m$

**sol.** Mass is

$$\begin{aligned} M &= \int_0^1 \int_{x^2}^{x} x\, dydx = \int_0^1 x\,[y]_{y=x^2}^{y=x}\, dx \\ &= \int_0^1 (x^2 - x^3)dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12} \end{aligned}$$

and $M_x$ is

$$\begin{aligned} M_x &= \int_0^1 \int_{x^2}^{x} yxdydx = \int_0^1 x\left[\frac{y^2}{2}\right]_{y=x^2}^{y=x} dx \\ &= \int_0^1 \frac{x}{2}(x^2 - x^4)dx = \left[\frac{x^4}{8} - \frac{x^6}{12}\right]_0^1 = \frac{1}{24}. \end{aligned}$$

□

When the density $\delta = 1$, the center of mass is also called the **centroid**.

**Example 14.6.3.** Find the centroid of the region bounded by $y = x$, $y = x^2$.

**sol.**

$$\begin{aligned} M &= \int_0^1 \int_{x^2}^{x} 1dydx = \int_0^1 [y]_{x^2}^{x}\, dx = \int_0^1 (x - x^2)dx = \frac{1}{6}, \\ M_x &= \int_0^1 \int_{x^2}^{x} ydydx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{x} dx = \int_0^1 (\frac{x^2}{2} - \frac{x^4}{2})dx = \frac{1}{15}, \\ M_y &= \int_0^1 \int_{x^2}^{x} xdydx = \int_0^1 x\,[y]_{x^2}^{x}\, dx = \int_0^1 (x^2 - x^3)dx = \frac{1}{12}. \end{aligned}$$

Hence

$$\bar{x} = \frac{1/12}{1/6} = \frac{1}{2}, \quad \bar{y} = \frac{1/15}{1/6} = \frac{2}{5}.$$

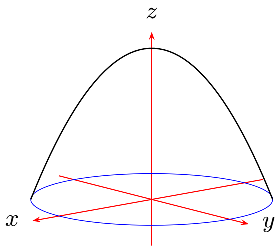



Figure 14.30: Region of $0 \leq z \leq 4 - x^2 - y^2$

$\square$

## Moment in 3 D

For 3-D, we have

**Definition 14.6.4** (Moment, center of mass)**.** Let $\delta(x, y, z)$ be the density of some material occupying some region $R$ in $\mathbb{R}^3$. Then

$$
\begin{aligned}
\text{The mass } M &= \iint \delta(x, y, z) dx dy dz, \\
\text{The moment w.r.t } yz\text{- plane } M_{yz} &= \iiint_R x\delta(x, y, z)\, dx dy dz, \\
\text{The moment w.r.t } xz\text{- plane } M_{xz} &= \iiint_R y\delta(x, y, z)\, dx dy dz, \\
\text{The moment w.r.t } xy\text{- plane } M_{xy} &= \iiint_R z\delta(x, y, z)\, dx dy dz.
\end{aligned}
$$

The **center of mass** $(\bar{x}, \bar{y}, \bar{z})$ is defined as

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}.$$

**Example 14.6.5.** Find the center of mass of a solid of constant density $\delta$ bounded by the disk $x^2 + y^2 \leq 4$ in the plan $z = 0$ and the paraboloid $z = 4 - x^2 - y^2$. (Fig 14.30)

**sol.**

$$
\begin{aligned}
M &= \int_0^1 \int_{x^2}^{x} 1 dy dx = \int_0^1 [y]_{x^2}^{x} \, dx = \int_0^1 (x - x^2) dx = \frac{1}{6}, \\
M_x &= \int_0^1 \int_{x^2}^{x} y dy dx = \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{x} \, dx = \int_0^1 (\frac{x^2}{2} - \frac{x^4}{2}) dx = \frac{1}{15}, \\
M_y &= \int_0^1 \int_{x^2}^{x} x dy dx = \int_0^1 x \, [y]_{x^2}^{x} \, dx = \int_0^1 (x^2 - x^3) dx = \frac{1}{12}.
\end{aligned}
$$

Hence

$$\bar{x} = \frac{1/12}{1/6} = \frac{1}{2}, \quad \bar{y} = \frac{1/15}{1/6} = \frac{2}{5}.$$

□

## Moment of inertia

Assume a mass is occupying the region $R$. The moment of inertia w.r.t a line $L$ is

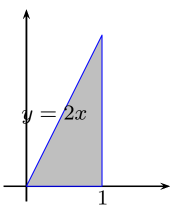
$$I_L = \iiint_R r^2(x, y, z) \delta(x, y, z) \, dx dy dz,$$

where $r(x, y, z)$ is the distance from the line $L$ to the point. If $L$ is the $x$ (or $y$, $z$) axis, the moment of inertia w.r.t a line $x$-axis is (resp.)

$$I_x = \iiint_R (y^2 + z^2) \delta \, dV, \ I_y = \iiint_R (x^2 + z^2) \delta \, dV, \ I_z = \iiint_R (x^2 + y^2) \delta \, dV.$$

**Example 14.6.6.** A thin plate is covering the triangular region bounded by $x$ axis, $x = 1$, $y = 2x$. The plate density is $\delta = 6(x + y + 1)$. Find the moment of inertia of the plate about the coordinate axes and the origin.

**sol.**




$$
\begin{aligned}
I_x &= \int_0^1 \int_0^{2x} y^2 \delta(x, y) dy dx \\
&= \textstyle\int_0^1 \int_0^{2x} 6y^2 (x + y + 1) dy dx \\
&= \int_0^1 \left[2xy^3 + \frac{3}{2} y^4 + 2y^3\right]_{y=0}^{y=2x} \, dx \\
&= \int_0^1 (40x^4 + 16x^3) \, dx \\
&= \left[8x^5 + 4x^4\right]_0^1 = 12.
\end{aligned}
$$

$I_y, I_0$ can be similarly obtained.

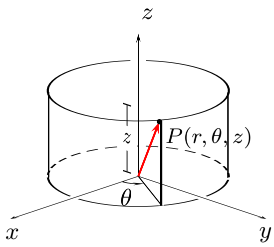
□

## 14.7 Triple integrals in Cylindrical and Spherical Coordinate

### Cylindrical coordinate system

Given a point $P = (x, y, z)$, we can use polar coordinate for $(x, y)$-plane. Then it holds that

$$\text{Cylindrical to Cartesain} \begin{cases} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z. \end{cases}$$




We say $(r, \theta, z)$ is **cylindrical coordinate** of $P$.

The expression $(r, \theta, z)$ is not unique.

**Example 14.7.1.** The set of all points $r = a$ in cylindrical coordinate is

$$\{(x, y, z) \mid x^2 + y^2 = a^2\}.$$

This is a cylinder (Figure 14.31).

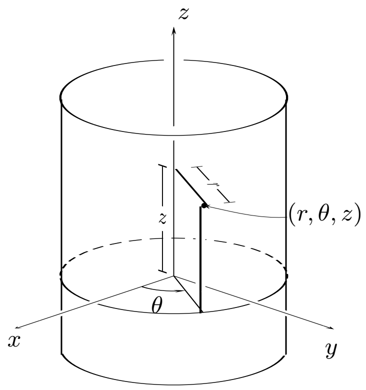
**Example 14.7.2.** The equation $r = 3\cos\theta$ gives

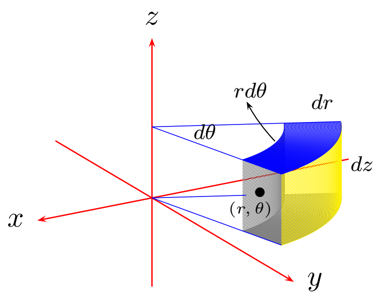
$$r^2 = 3r\cos\theta \Rightarrow x^2 + y^2 = 3x.$$

This holds for all $z$. This is again a cylinder.

**Example 14.7.3.** Identify the surface given by the equation $z = 2r$ in cylindrical coordinate.

**sol.** Squaring, we have $z^2 = 4r^2 = 4(x^2 + y^2)$. The section $z = c$ is $c^2 = 4(x^2 + y^2)$, while with $x = 0$ we have $z = \pm y$. With $y = 0$ we have $z = \pm x$. Thus this is a cone.




A sector of a cylinder

Figure 14.31: cylindrical coordinate

□

**Example 14.7.4.** Change the equation $x^2 + y^2 - z^2 = 1$ to cylindrical coordinate.

**sol.** $r^2 - z^2 = 1.$

□

### 14.7.1 Integration in Cylindrical Coordinate

Let $D$ be any region in $\mathbb{R}^3$. We describe it using the coordinate

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

We partition the region $D$ into small cylindrical wedges (Fig 14.31); Small wedge given by

$$[r_k, r_k + \Delta r_k] \times [\theta_k, \theta_k + \Delta\theta_k] \times [z_k, z_k + \Delta z_k]$$

has volume $\Delta V_k = \Delta A_k \Delta z_k \doteq r_k \Delta r_k \Delta\theta_k \Delta z_k$. So the sum $\sum_k f(x_k, y_k, z_k)\Delta V_k$ approaches

$$\iiint_D f(x, y, z)\, dxdydz = \iiint_{D^*} f(r\cos\theta, r\sin\theta, z) r\, dzdrd\theta. \qquad (14.13)$$

Here $D^*$ is the region of described by the cylindrical coordinate $(r, \theta, z)$.

How to integrate in cylindrical coordinate? (iterated integrals)

(1) Sketch the region $D$ and its projection $R$ on the $xy$ plane.

(2) Find the $z$ limits of integration: $g_1(r,\theta) \le z \le g_2(r,\theta)$.

(3) Find the $r$ limits of integration : $h_1(\theta) \le r \le h_2(\theta)$.

(4) Find the $\theta$ limits of integration : $\alpha \le \theta \le \beta$ and set

$$\iiint_D f(x,y,z)\,dxdydz = \int_\alpha^\beta \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r,\theta,z)dzdrd\theta.$$

**Example 14.7.5.** Find the centroid of the solid bounded by the cylinder $x^2+y^2=4$ between $0 \le z \le x^2+y^2$. Sol. We need $(\bar{x},\bar{y},\bar{z})$. By symmetry we have $\bar{x}=\bar{y}=0$. To find $\bar{z}=\frac{M_{xy}}{M}$ we need to compute

$$M = \iiint_{D^*} dz\,rdrd\theta, \quad M_{xy} = \iiint_{D^*} zdz\,rdrd\theta.$$

(1) Sketch the region.

(2) Find the $z$ limits of integration : $0 \le z \le x^2+y^2 = r^2$.

(3) Find the $r$ limits of integration : $0 \le r \le 2$.

(4) Find the $\theta$ limits of integration : $0 \le \theta \le 2\pi$.

$$M = \int_0^{2\pi}\int_0^2\int_0^{r^2} dz\,rdrd\theta = 8\pi, \quad M_{xy} = \int_0^{2\pi}\int_0^2\int_0^{r^2} zdz\,rdrd\theta = \frac{32\pi}{3}.$$

Therefore

$$\bar{z} = \frac{M_{xy}}{M} = \frac{4}{3}.$$

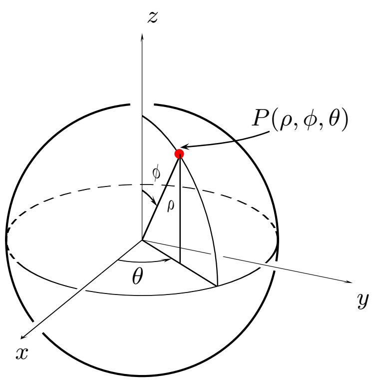
### 14.7.2 Integration in spherical coordinate system

We call $(\rho,\phi,\theta)$ to be the **spherical coordinate** of $P(x,y,z)$ if

(1) $\rho$ is the distance from $P$ to the origin

(2) $\phi$ is the angle that makes with positive $z$ axis

(3) $\theta$ is the angle from cylindrical coordinate.



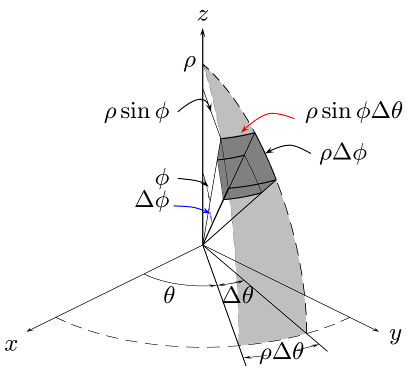




Figure 14.32: Spherical coordinate

For the point $P(x, y, z)$ we have

| | | |
|---|---|---|
| Spherical to Cartesian | $\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$ | $\begin{pmatrix} \rho \geq 0 \\ 0 \leq \theta < 2\pi \\ 0 \leq \phi \leq \pi \end{pmatrix}$ |

**Example 14.7.6.** (1) Find the spherical coord. of $(1, -1, 1)$.

(2) Find the cartesian coord. of $(3, \pi/6, \pi/4)$.

(3) Find the spherical coord. of $(2, -3, 6)$.

(4) Find the spherical coord. of $(-3, -3, \sqrt{6})$.

**sol.** (1) $\rho = \sqrt{3}$.

$$\phi = \cos^{-1}(\frac{z}{\rho}) = \cos^{-1}(\frac{1}{\sqrt{3}}) \approx 0.955 \approx 54.74^o.$$

Since the point $(1, -1)$ lies in the 4-th quadrant, we see

$$\theta = 2\pi + \arctan(\frac{y}{x}) = 2\pi + \arctan(-1) = \frac{7\pi}{4}.$$

(3) $\rho = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{2^2 + (-3)^2 + 6^2} = 7.$

$$\phi = \cos^{-1}(\frac{z}{\rho}) = \cos^{-1}\left(\frac{6}{7}\right).$$

To find $\theta$, we see $\tan\theta = -3/2$. Since the point lies in the fourth quadrant, we have

$$\theta = 2\pi + \tan^{-1}(-3/2).$$

(4)

$$\begin{aligned}
\rho &= \sqrt{9+9+6} = 2\sqrt{6} \\
\phi &= \cos^{-1}\left(\frac{\sqrt{6}}{2\sqrt{6}}\right) = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3} \\
\theta &= \pi + \tan^{-1}(\frac{-1}{-1}) = \pi + \frac{\pi}{4} = \frac{5\pi}{4}(\text{ 3rd quadrant}).
\end{aligned}$$

Hence the spherical coordinate is $(2\sqrt{6}, \pi/3, 5\pi/4)$.

□

**Example 14.7.7.** Express the surface $x^2 + y^2 + (z-1)^2 = 1$ using spherical coordinate.

$$\rho^2 - 2\rho\cos\phi + 1 = 1 \to \rho 2 = 2\phi.$$

**Example 14.7.8.** Express the cone $x = x^2 + y^2$ using spherical coordinate.

$$\rho\cos\phi = \rho\sin\phi \to \phi = \frac{\pi}{4}.$$

**Example 14.7.9.** Express the surface (1) $xz = 1$ and (2) $x^2 + y^2 - z^2 = 1$ in spherical coordinate.

**sol.** (1) Since $xz = \rho^2 \sin\phi\cos\theta\cos\phi = 1$, we have the equation

$$\rho^2 \sin 2\phi \cos\phi = 2.$$

(2) Since $x^2+y^2-z^2 = x^2+y^2+z^2-2z^2 = \rho^2-2(\rho\cos\phi)^2 = \rho^2(1-2\cos^2\phi)$, the equation is $\rho^2(1-2\cos^2\phi) = 1$.

□

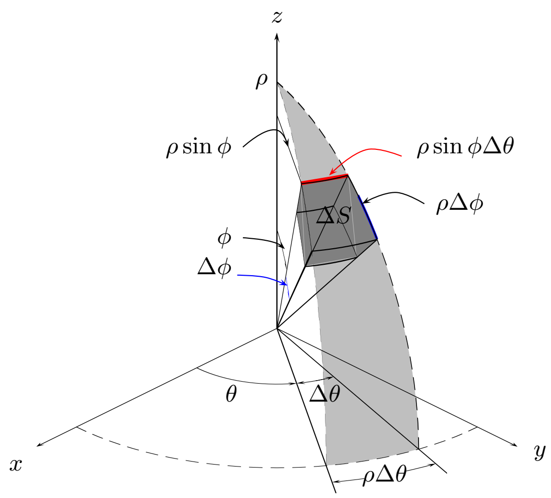



Figure 14.33: Partition in spherical coordinate

## Volumes in Spherical Coordinate-Geometric Derivation

Consider the small region bounded by the following conditions: (Fig.14.33)

$$\rho_0 \leq \rho \leq \rho_0 + \Delta\rho, \quad \phi_0 \leq \phi \leq \phi_0 + \Delta\phi, \quad \theta_0 \leq \theta \leq \theta_0 + \Delta\theta.$$

The region is between two spheres of radius $\rho_0$, $\rho_0 + \Delta\rho$, two cones $\phi = \phi_0$, $\phi = \phi_0 + \Delta\phi$ and two planes $\theta = \theta_0$, $\theta = \theta_0 + \Delta\theta$.

**Remark 14.7.10.** This region corresponds to a 'box' in $(\rho, \phi, \theta)$ coordinate:

$$[\rho_0, \rho_0 + \Delta\rho] \times [\phi_0, \phi_0 + \Delta\phi] \times [\theta_0, \theta_0 + \Delta\theta].$$

First let us find the area of the region $\Delta S$ bounded by $\theta_0 \leq \theta \leq \theta_0 + \Delta\theta$, $\phi_0 \leq \phi \leq \phi_0 + \Delta\phi$ on the sphere of radius $\rho$. The distance from a point on the surface to the $z$-axis is $\rho \sin\phi$. When $\Delta\rho$ and $\Delta\theta$ are small, this rectangular like region on the sphere can be approximated by rectangle whose base is $\rho \sin\phi\Delta\theta$, height is $\rho\Delta\phi$:

$$\Delta S \approx \rho^2 \sin\phi\Delta\phi\Delta\theta.$$

Now consider the solid formed by this region with thickness $\Delta\rho$. Then its volume is

$$\Delta V \approx \rho^2 \sin\phi \Delta\rho\Delta\phi\Delta\theta.$$

Hence, as in the early discussions, the volume of $D$ is defined as

$$\iiint_D dV = \int\int\int \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \tag{14.14}$$

and for a continuous function $f$ on $D$, the integral is defined as

$$\iiint_D f dV = \int\int\int f(\rho,\phi,\theta)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta. \tag{14.15}$$

**How to integrate in Spherical coordinates**

Let $D$ be the region determined by

$$D = \{(\rho,\phi,\theta) : g_1(\phi,\theta) \le \rho \le g_2(\phi,\theta), h_1 \le \phi \le h_2, \alpha \le \theta \le \beta\}.$$

To evaluate $\iiint_D f dV = \int\int\int f(\rho,\phi,\theta)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ we proceed as follows:

(1) Sketch the region $D$ and project it onto $xy$ plane.

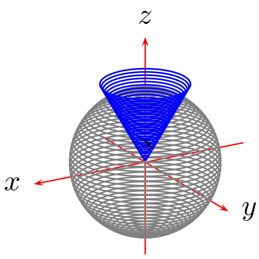
(2) Find the $\rho$ limit of the integration $(g_1(\phi,\theta) \le \rho \le g_2(\phi,\theta))$

(3) Find the $\phi$ limit of the integration $(h_1(\theta) \le \phi \le h_2(\theta))$

(4) Find the $\theta$ limit of the integration

**Example 14.7.11.** Find the volume of the "ice cream cone" $D$ cut from the solid $\rho \le 1$ by the cone $\phi = \pi/3$.

sol.




$\rho \leq 1, \ \phi \leq \pi/3$

$$\begin{aligned} V &= \iiint_D \rho^2 \sin\phi d\rho\, d\phi\, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_0^1 \sin\phi\, d\phi\, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi\, d\phi\, d\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{3}\cos\phi\right]_0^{\pi/3} d\theta \\ &= 2\pi\left(-\frac{1}{6}+\frac{1}{3}\right) = \frac{\pi}{3}. \end{aligned}$$

□

**Example 14.7.12.** Compute the moment of inertia of about $z$-axis of the solid occupying the same region as above with density $\delta = 1$.

sol. The moment of inertia is by spherical coordinate,

$$I_z = \iiint_D (x^2+y^2) dV = \iiint_D \rho^2 \sin^2\phi \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

Changing it to an iterated integral, we have

$$\begin{aligned} I_z &= \iiint_D \rho^4 \sin^3\phi d\rho\, d\phi\, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^4 \sin^3\phi d\rho\, d\phi\, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \left[\frac{\rho^5}{5}\right]_0^1 \sin^3\phi\, d\phi\, d\theta \\ &= \frac{1}{5}\int_0^{2\pi} \int_0^{\pi/3} (1-\cos^2\phi)\sin\phi\, d\phi\, d\theta \\ &= \frac{1}{5}\int_0^{2\pi} \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/3} d\theta \\ &= \frac{1}{5}\int_0^{2\pi} \left(-\frac{1}{2}+1+\frac{1}{24}-\frac{1}{3}\right) d\theta = \frac{\pi}{12}. \end{aligned}$$

□

**Example 14.7.13.** Compute

$$\iiint_W \exp(x^2+y^2+z^2)^{3/2} dV,$$

where $W$ is the unit ball.

sol. By spherical coordinate,

$$\iiint_W \exp(x^2+y^2+z^2)^{3/2} dV = \iiint_{W^*} \rho^2 e^{\rho^3} \sin\phi d\theta\, d\phi\, d\rho.$$

Changing it to an iterated integral, we have

$$\begin{aligned} & \int_0^1 \int_0^\pi \int_0^{2\pi} \rho^2 e^{\rho^3} \sin\phi d\theta\, d\phi\, d\rho \\ = & \ 2\pi \int_0^1 \int_0^\pi \rho^2 e^{\rho^3} \sin\phi d\phi\, d\rho \\ = & \ 4\pi \int_0^1 \rho^2 e^{\rho^3} d\rho = \frac{4}{3}\pi(e-1). \end{aligned}$$

□

## 14.8 Substitution-Change of variables

We recall one variable case: If $x : [a,b] \to [c,d]$ is $\mathcal{C}^1$ function and $f : [c,d] \to \mathbb{R}$ is integrable, then the integral of $f$ on $[c,d]$ can be changed to an integral over $[a,b]$ by

$$\int_c^d f(x)dx = \int_a^b f(x(t))x'(t)dt. \tag{14.16}$$

Here the change in the integrand is just the linear scaling factor $x'(t)$ and the change in the domain is again a linear scaling to $[a,b]$. But for functions with two or more variables, the situation is not so simple, because the shape of domain change nontrivially.

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(x(u,v),y(u,v))(\text{ some factor })dudv \tag{14.17}$$

Let $F(u,v) = f(x(u,v),y(u,v))$ and recalling the definition of integral, we see

$$\lim_{n\to\infty} \sum_{i=1}^n f(x_i,y_i)\Delta A_i(x,y) = \lim_{n\to\infty} \sum_{i=1}^n F(u_i,v_i)\Delta A_i(u,v). \tag{14.18}$$

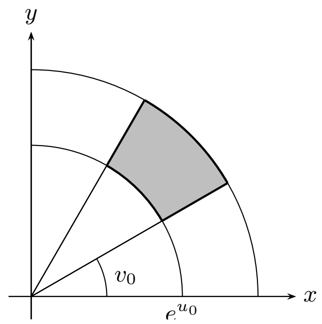



Figure 14.34: Inverse image of a polar rectangle

### One-to-one map and onto map

**Definition 14.8.1.** A map $T$ is called **one to one** on $D^*$, if for $(u, v)$ and $(u', v') \in D^*$, $T(u, v) = T(u', v')$ implies $(u, v) = (u', v')$.

**Example 14.8.2.** Show the polar coordinate map $T = T(r, \theta) = (r\cos\theta, r\sin\theta)$ is not one-to-one. But it is so if we restrict to the region $0 \le \theta < 2\pi$.

**Definition 14.8.3.** A map $T$ is called **onto** $D$, if for every point $(x, y) \in D$ there exists at least a point $(u, v) \in D$ such that $T(u, v) = (x, y)$.

Thus if $T$ is onto then we can solve the equation $T(u, v) = (x, y)$. If, in addition, $T$ is one-to-one, the solution is unique.

**Example 14.8.4.** A linear transform $A\mathbf{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ given by a matrix $A$ is **one to one** and **onto** if $\det A \neq 0$.

**Example 14.8.5.** Let $D$ be the region in the first quadrant lying between concentric circles $r = a, r = b$ and $\theta_1 \le \theta \le \theta_2$. (Fig. 14.34) Let

$$T(r, \theta) = (r\cos\theta, r\sin\theta)$$

be the polar coordinate map. Find a region $D^*$ in $(r, \theta)$ coordinate plane such that $D = T(D^*)$.

**sol.** In $D$, we see

$$a \le r \le b, \quad \theta_1 \le \theta \le \theta_2.$$

Hence

$$D^* = [a, b] \times [\theta_1, \theta_2].$$

□

## Coordinate transformations

Let $D^*$ be a region in $\mathbb{R}^2$. Suppose $T$ is $C^1$-map $D^* \to \mathbb{R}^2$. We denote the image by $D = T(D^*)$. (Fig 14.35)

$$T(D^*) = \{(x, y) \mid (x, y) = T(u, v), \quad (u, v) \in D^*\}.$$

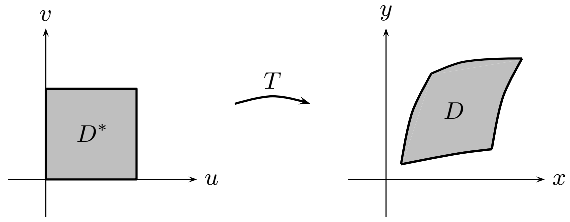



Figure 14.35: The transformation $T$ maps $D^*$ to $D$

**Example 14.8.6.** Let $D^*$ be the rectangle $D^* = [0, 1] \times [0, 1]$ in $(u, v)$ plane. Find the image of $D^*$ under $T = T(u, v) = (u + \frac{v}{2}, \frac{u}{3} + v)$ and the area.

**sol.** The image is a parallelogram (Figure 14.40) formed by two vectors $(1, \frac{1}{2}), (\frac{1}{3}, 1)$ and the area is

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| \equiv \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & 1 \end{vmatrix} = \frac{5}{6}.$$

We remark that these vectors are two columns of the derivative $DT$.

□

From this example it is not difficult to guess the following change of variable form:

$$\iint_D f(x, y)dxdy = \iint_{D^*} f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv \qquad (14.19)$$

which is correct when $f = constant$ and $T$ is a linear map. Indeed, this is true in general, we will see it soon.

## Jacobian Determinant-measures change of area

We first see how the area of a region changes under a linear map. Let $D^* =$

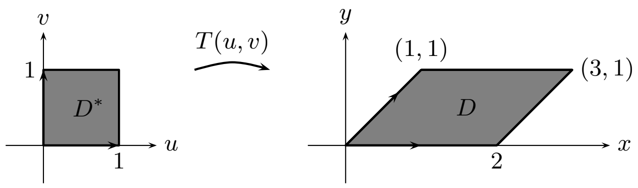



Figure 14.36: The image of a rectangle under a linear transform $T$

$[0,1]\times[0,1]$, and construct a linear map $T$ that maps $D^*$ onto a parallelogram $D$ (See Figure 14.40). Consider the vector $\mathbf{c}_1 := \mathbf{a}_2 - \mathbf{a}_1$, $\mathbf{c}_2 := \mathbf{a}_4 - \mathbf{a}_1$, and set (one may assume $\mathbf{a}_1 = 0$)

$$T(u,v) = \mathbf{c}_1 u + \mathbf{c}_2 v + \mathbf{a}_1 = \mathbf{c}_1 u + \mathbf{c}_2 v.$$

Then we can check $T(u,0)$ maps the line segment $\{0 \le u \le 1, v = 0\}$ to the side $\overline{\mathbf{a}_1\mathbf{a}_2}$. Similarly, $T(0,v)$ maps the line segment $\{0 \le v \le 1, u = 0\}$ to the side $\overline{\mathbf{a}_1\mathbf{a}_4}$. Hence we conclude $T$ is the desired map. The two tangent vectors to $D$ at the origin are

$$\begin{aligned} T_u &= \mathbf{a}_2 - \mathbf{a}_1 \\ T_v &= \mathbf{a}_4 - \mathbf{a}_1. \end{aligned}$$

The area of the parallelogram $D$ is $\|(\mathbf{a}_2 - \mathbf{a}_1) \times (\mathbf{a}_4 - \mathbf{a}_1)\|$ (viewed as three dimensional vectors) But this is nothing but the absolute value of the determinant of the derivative of $T$. Thus

$$Area(D) = |J|,$$

where

$$J = \frac{\partial(x,y)}{\partial(u,v)} := det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = |DT|.$$

$J$ is called the **Jacobian of** $T$. Hence we see the rectangle of dimensions $\Delta u$, $\Delta v$ along $u$, $v$ direction is mapped to a parallelogram with area $|J|\Delta u \Delta v$.

Thus for the area change, we have

**Theorem 14.8.7.** *Let $A$ be a $2\times 2$ matrix with non zero determinant. Let $T$ be a linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Then $T$ maps a parallelogram $D^*$ onto the parallelogram $D = T(D^*)$ and*

$$\textit{Area of } D = |\textit{det } A| \cdot (\textit{Area of } D^*).$$

**Remark 14.8.8.** (1) A similar statement holds for a linear map with nonzero determinant from $\mathbb{R}^3$ to $\mathbb{R}^3$.

**Example 14.8.9.** Let $T$ be $((x+y)/2, (x-y)/2)$ and let $D$ be the square whose vertices are $(1,0), (0,1), (-1,0), (0,-1)$. Find a $D^*$ such that $D = T(D^*)$.

**sol.** Since $T$ is linear $T(\mathbf{x}) = A\mathbf{x}$ where $A$ is $2 \times 2$ matrix whose determinant is nonzero. $T^{-1}$ is also a linear transform. Hence by Theorem 14.8.7, $D^*$ must be a parallelogram. To find $D^*$, it suffices to find the inverse image of vertices. It turns out that

$$D^* = [-1,1] \times [-1,1].$$

Now

$$A(D) = (\sqrt{2})^2 = 2, \ |\textit{det } A| = \frac{1}{2}, \ A(D^*) = 4, .$$

□

This idea can be generalized to non-linear mappings.

## Change of variable in the definite integrals

Given two regions $D$ and $D^*$, a differentiable mapping $T$ on $D^*$ with image $D = T(D^*)$, we would like to express the integral $\iint_D f(x,y)dxdy$ as an integral over $D^*$ of the composite function $f \circ T$. We write $T$ as

$$T(u,v) = (x(u,v), y(u,v)) \text{ for } (u,v) \in D^*.$$

Then we have

$$\iint_D f(x,y)\,dxdy \quad = \quad \iint_{D^*} f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv. \qquad (14.20)$$

As a special case, when $f = 1$, we obtain the area

$$\iint_D dxdy \quad = \quad \iint_{D^*} |J|\,dudv = \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv. \qquad (14.21)$$

**Example 14.8.10.** Let $D^*$ be the rectangle $D^* = [0,1] \times [0,\pi/3]$ in $(r,\theta)$ plane. Find the image of $D^*$ under $T = T(r,\theta) = (r\cos\theta, r\sin\theta)$.

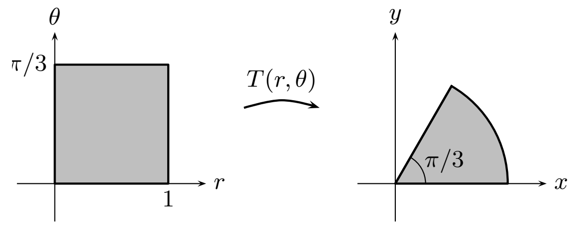



Figure 14.37: Map by polar coordinate

**sol.** Let $T(r,\theta) = (x,y)$. Then $x^2+y^2 = r^2$, $0 \le r \le 1$. Thus $D$ is the circular sector $0 \le r \le 1$, $0 \le \theta \le \pi/3$. (Figure 14.37)

□

**Example 14.8.11.** Evaluate

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dydx.$$

**sol.** Let us use the substitution $u = x+y, v = y-2x$, so that

$$x = \frac{u}{3} - \frac{v}{3}, \quad y = \frac{2u}{3} + \frac{v}{3}. \tag{14.22}$$

One can find the limits of integration and find $J(u,v) = \frac{1}{3}$. To find the limit of integration, we see Figure 14.38. and Table 14.1.

Table 14.1: Limit of integration for Example 14.8.11

| $xy$ eq. for boundary | $uv$ eq. for boundary | Simplified |
|---|---|---|
| $x+y=1$ | $\frac{u-v}{3} + \frac{2u+v}{3} = 0$ | $u = 1$ |
| $x = 0$ | $\frac{u}{3} - \frac{v}{3} = 0$ | $v = u$ |
| $y = 0$ | $\frac{2u+v}{3} = 0$ | $v = -2u$ |

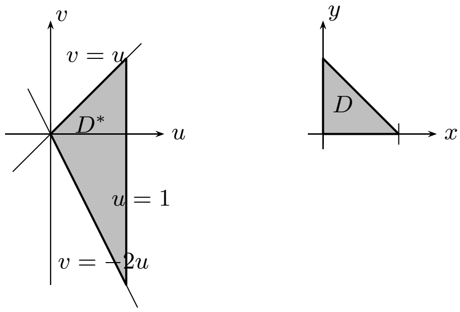



Figure 14.38: Change of variables for Example 14.8.11

Hence we obtain

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dydx &= \int_0^1 \int_{v=-2u}^{v=u} \sqrt{u}v^2 |J(u,v)| dvdu \\
&= \frac{1}{3}\int_0^1 \sqrt{u}\left[\frac{v^3}{3}\right]_{-2u}^{u} du \\
&= \frac{1}{9}\int_0^1 \sqrt{u}(u^3+8u^3)du \\
&= \int_0^1 u^{7/2}du = \frac{2}{9}.
\end{aligned}$$

□

**Example 14.8.12.** Evaluate

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dxdy.$$

**sol.** We use the substitution $u = \sqrt{xy}, v = \sqrt{\frac{y}{x}}$, so that

$$x = \frac{u}{v}, \quad y = uv, u, v > 0. \tag{14.23}$$

We see

$$J(u,v) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}.$$

(Note that if we integrate w.r.t $u$ first, we run into trouble!) Once we find

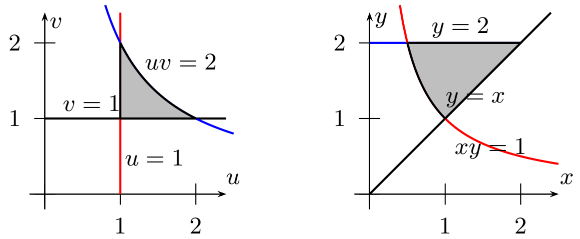



Figure 14.39: Change of variables for Example 14.8.12

Table 14.2: Limit of integration for Example 14.8.12

| $xy$ eq. for boundary | $uv$ eq. for boundary | Simplified |
|---|---|---|
| $y = x$ | $uv = \frac{u}{v}$ | $v = 1 (v > 0)$ |
| $xy = 1$ | $u = 1$ | $u = 1$ |
| $y = 2$ | $u = \sqrt{2x},\ v = \sqrt{\frac{2}{x}}$ | $uv = 2$ |

the limits of integration(need the region $D$ and $D^*$) from Table 14.2, we obtain

$$\begin{aligned}
\iint_R \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dxdy &= \iint_R ve^u \frac{2u}{v} dudv \\
&= \int_1^2 \int_1^{2/u} 2ue^u dvdu \\
&= 2 \int_1^2 \left[vue^u\right]_{v=1}^{v=2/u} du \\
&= 2 \int_0^1 (2e^u - ue^u) du \\
&= 2\left[(2e^u - ue^u) + e^u\right]_{u=1}^{u=2} = 2e(e-2).
\end{aligned}$$

□

## Change of variable formula - general case

Above idea of computing area of $D = T(D^*)$ can used when $T$ is a differentiable (nonlinear) mapping from a subset of $\mathbb{R}^2$ to $\mathbb{R}^2$ by using the linear(tangent plane) approximation of $T$. Let $D^* = [u_0, u_0 + \Delta u] \times [v_0, v_0 + \Delta v]$ and $D$ be the image of $D^*$ under $T$. Consider

$$T(u,v) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(u_0,v_0) + \frac{\partial x}{\partial u}(u_0,v_0)\Delta u + \frac{\partial x}{\partial v}(u_0,v_0)\Delta v + h.o.t \\ y(u_0,v_0) + \frac{\partial y}{\partial u}(u_0,v_0)\Delta u + \frac{\partial y}{\partial v}(u_0,v_0)\Delta v + h.o.t \end{bmatrix} \tag{14.24}$$

In vector form, we have

$$T\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{X} = \mathbf{X}_0 + DT\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} + h.o.t$$

and replace the map $T$ by its linear part $DT$.

## Geometric meaning of $DT$

Let

$$T_u := DT(u,v)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{bmatrix}$$

and

$$T_v := DT(u,v)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{bmatrix}.$$

First we note that the two curves $T(\cdot,v)$ and $T(u,\cdot)$ describes the boundary of $D = T(D^*)$ at $T(u,v)$. First fix the $v$ variable. Then $T_u$ is a tangent vector to the curve $T(u,v)$(as a function of $u$). Similarly, for each fixed $u$, $T(u,v)$ represents a curve with $v$ as a parameter. Hence $T_v$ is a tangent vector to the curve $T(u,v)$.

Now the two tangent vectors

$$T_u\Delta u, \quad T_v\Delta v$$

form a parallelogram approximating the region $D$(Figure 14.40). Hence the area of the parallelogram is (the absolute value of)

$$\begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\Delta u\Delta v = \frac{\partial(x,y)}{\partial(u,v)}\Delta u\Delta v \doteq J \cdot A(D^*).$$

$$\text{In other words} \quad \|T_u \times T_v\|\Delta u\Delta v = |J|\Delta u\Delta v.$$

Summing over all subregions and taking the limit as $\Delta u, \Delta v \to 0$ we obtain the formula (14.21), (14.20).

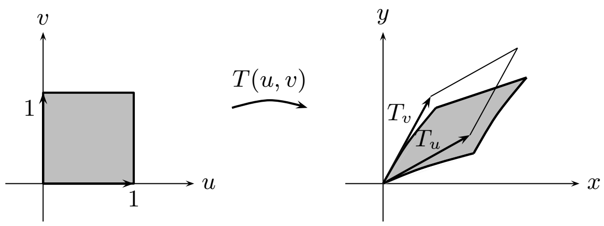



Figure 14.40: approximate $T(D^*)$

## Change of Variables in Triple Integrals

**Definition 14.8.13.** Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w)).$$

The the **Jacobian** $J$ is again, as 2D case, the determinant of the derivative $DT$

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u}, & \frac{\partial x}{\partial v}, & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u}, & \frac{\partial y}{\partial v}, & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u}, & \frac{\partial z}{\partial v}, & \frac{\partial z}{\partial w} \end{bmatrix}.$$

The absolute value of this determinant is equal to the volume of parallelepiped determ'd by the following vectors

$$\begin{aligned} \mathbf{T}_u &= \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \\ \mathbf{T}_v &= \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} \\ \mathbf{T}_w &= \frac{\partial x}{\partial w}\mathbf{i} + \frac{\partial y}{\partial w}\mathbf{j} + \frac{\partial z}{\partial w}\mathbf{k}, \end{aligned}$$

which is the absolute value of the triple product (recall Chap. 12.4)

$$|(\mathbf{T}_u \times \mathbf{T}_v) \cdot \mathbf{T}_w| = |J|.$$

Caution: Three vectors $\mathbf{T}_u, \mathbf{T}_v, \mathbf{T}_w$ are column vectors of $DT$, but since $det(A) = det(A^T)$ for any square matrix, we have

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} \mathbf{T}_u, & \mathbf{T}_v, & \mathbf{T}_w \end{bmatrix}.$$

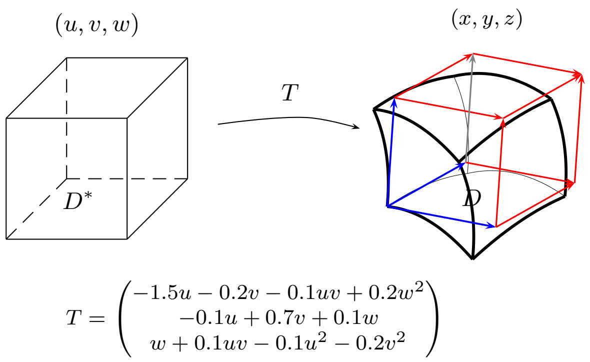



Figure 14.41: Deformed box and parallelepiped generated by tangent vectors.

**Theorem 14.8.14.** *If $T$ is a $\mathcal{C}^1$- map from $D^*$ onto $D$ in $\mathbb{R}^3$ and $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous, then*

$$\iiint_D dxdydz = \iiint_{D^*} |J|\, dudvdw, \tag{14.25}$$
$$\iiint_D f(x,y,z)\, dxdydz = \iiint_{D^*} f(T(u,v,w))|J|\, dudvdw. \tag{14.26}$$

**Example 14.8.15.** Evaluate

$$\int_0^3 \int_0^4 \int_{y/2}^{y/2+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dxdydz$$

using the transformation

$$u = (2x-y)/2, \quad v = y/2, \quad w = z/3. \tag{14.27}$$

**sol.** We see

$$x = u + v, \quad y = 2v, \quad z = 3w. \tag{14.28}$$

We see

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u}, & \frac{\partial x}{\partial v}, & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u}, & \frac{\partial y}{\partial v}, & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u}, & \frac{\partial z}{\partial v}, & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

One can find the limits of integration we obtain

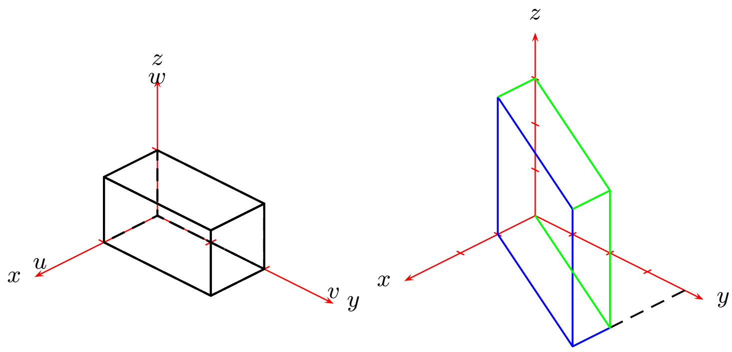



Figure 14.42: Transformation in Example 14.8.15

Table 14.3: Limit of integration for Example 14.8.15

| $xyz$ eq. for boundary | $uvw$ eq. for boundary | Simplified eq. |
|---|---|---|
| $x = y/2$ | $u + v = 2v/2$ | $u = 0$ |
| $x = y/2 + 1$ | $u + v = 2v/2 + 1$ | $u = 1$ |
| $y = 0$ | $2v = 0$ | $v = 0$ |
| $y = 4$ | $2v = 4$ | $v = 2$ |
| $z = 0$ | $3w = 0$ | $w = 0$ |
| $z = 3$ | $3w = 3$ | $w = 1$ |

$$\begin{aligned}
\iiint_D f\,dxdydz &= \int_0^1 \int_0^2 \int_0^1 (u+w)\,|J|dudvdw \\
&= 6\int_0^1 \int_0^2 \left[\frac{u^2}{2} + uw\right]_0^1 dvdw \\
&= 6\int_0^1 \int_0^2 \left(\frac{1}{2} + w\right) dvdw \\
&= 6\int_0^1 (1+2w)\,dw = 12.
\end{aligned}$$

□

## Cyindrical Coordinate - revisited

Consider the cylindrical coordinate

$$(x(r,\theta,z), y(r,\theta,z), z(r,\theta,z)).$$

The Jacobian of the mapping $(r, \theta, z) \to (x, y, z)$ is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

$$\iiint_D f(x,y,z)\,dxdydz = \iiint_{D^*} F(r,\theta,z)dz\,rdr\,d\theta.$$

## Spherical Coordinate - revisited

**Example 14.8.16.** Derive the integration formula in spherical coordinate using Theorem 14.8.14.

**sol.** Spherical coordinate is given by

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.$$

The Jacobian of the mapping $(\rho, \phi, \theta) \to (x, y, z)$ is

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\ &= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\ &+ \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\ &= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) = \rho^2\sin\phi. \end{aligned}$$

Hence

$$\iiint_D f(x,y,z)\,dxdydz = \iiint_{D^*} F(\rho,\phi,\theta)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Here $F(\rho,\phi,\theta)$ means $f(x(\rho,\phi,\theta), y(\rho,\phi,\theta), z(\rho,\phi,\theta))$. This agrees with earlier formula (14.14), (14.15) derived by geometric intuition.

□

**Example 14.8.17.** The region $D$ is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1.$$

Find

$$\iiint_D |xyz| dxdydz.$$

**sol.** Let $T(u,v,w) = (au, bv, cw)$. Then $T$ maps the unit ball $D^* = \{(u,v,w) \mid u^2+v^2+w^2 \le 1\}$ to $D$ one-to-one, onto fashion. Since $J(T) = abc$ we have

$$\begin{aligned}\iiint_D |xyz| dxdydz &= \iiint_{D^*} (abc)^2 |uvw|\, du\, dv\, dw \\ &= 8 \iiint_{D^*_+} (abc)^2 uvw\, du\, dv\, dw.\end{aligned}$$

Here $D^*_+$ denotes the region $u \ge 0, v \ge 0, w \ge 0$ among $D^*$. Now use spherical coordinate,

$$\begin{aligned}& 8 \iiint_{D^*_+} (abc)^2 uvw\, du\, dv\, dw \\ =\;& 8(abc)^2 \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^5 \sin^3\phi \cos\phi \sin\theta\cos\theta\, d\rho\, d\phi\, d\theta \\ =\;& 8(abc)^2 \int_0^{\pi/2}\int_0^{\pi/2} \left[\frac{\rho^6}{6}\right]_0^1 \sin^3\phi\cos\phi\sin\theta\cos\theta\, d\phi\, d\theta \\ =\;& \frac{4}{3}(abc)^2 \int_0^{\pi/2} \left[\frac{\sin^4\phi}{4}\right]_0^{\pi/2} \sin\theta\cos\theta\, d\theta \\ =\;& \frac{1}{3}(abc)^2 \int_0^{\pi/2} \sin\theta\cos\theta d\theta \\ =\;& \frac{1}{3}(abc)^2 \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{1}{6}(abc)^2.\end{aligned}$$

□